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Games, rules, and solutions

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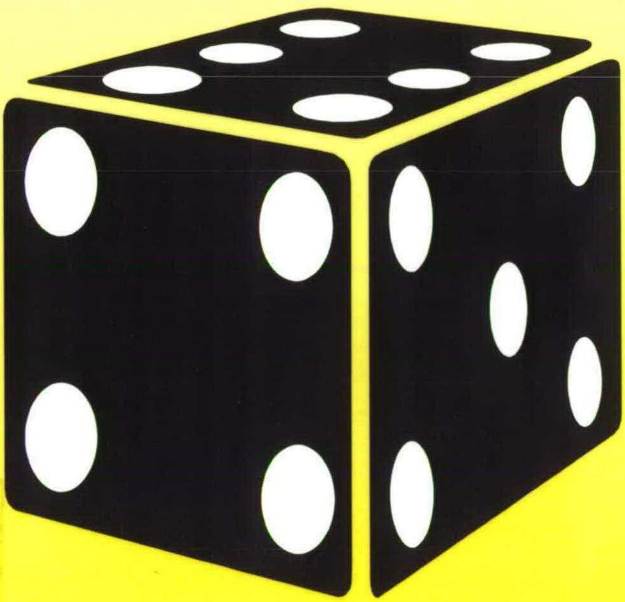
Games, Rules, and Solutions



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Games, Rules, and Solutions

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Proefschrift ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof.dr. L.F.W. de Klerk, in het openbaar te verdedigen ten overstaan van een door het college van dekanen aangewezen commissie in de aula van de Universiteit op vrijdag 24 februari 1995 te 16.15 uur door

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geboren op 30 september 1967 te Haps

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While writing this thesis I had the help and support of several people. First, I would like to thank Eric van Damme. This thesis would not have been written without his guidance and stimulance. I also thank him for co-authoring me on several occasions. I also would like to thank Helmut Bester, Tilman Börgers, Peter Borm, Hans Peters and Stef Tijs for joining the thesis committee. I benefitted from discussions with many colleagues and friends. In particular, I thank Paul de Bijl, Roland Strausz and Tonny Hurkens.

Finally, I would like to thank Thomas Schelling for writing his marvelous and inspiring book *The strategy of conflict.*

Contents

Chapter 1

Introduction

Games,...

Game theory deals with situations in which several persons with (possibly) diverging interests are involved. These persons (players) have to make choices, which then together determine the outcome. All players have some knowledge of the choices available to each player and of the preferences of each player regarding the outcomes. In their monumental book *Theory of games and economic behavior*, John von Neumann and Oskar Morgenstern (1944) argued that economic problems may be analyzed as games. Once all irrelevant details are stripped away from an economic problem, one is left with an abstract decision problem: a game.

Traditionally, game theory has been divided into a cooperative and a non-cooperative branch. In a cooperative game players are assumed to be able to make binding agreements and to form coalitions, although this is not explicitly shown by the rules of the game. Such a game is namely described by what each coalition can achieve when its members cooperate. The main questions addressed in cooperative game theory are which coalition will form and how the gains should be divided among its members. In this thesis only non-cooperative games will be studied. The rules of a non-cooperative game describe precisely which choices are available to each player. Players are only able to make binding agreements whenever the rules of the game permit them to do so. Each player has to decide which plan of action to follow in such a game, independently of the other players. Since the outcome of the game will depend on the actions of all players, which action is optimal for a player depends on the actions the other players choose. The action that a player chooses, therefore, depends on what he expects the other players to do.

..., *Rules*,...

In order to model an economic situation as a game one needs to answer the following questions. Who are the players? Which actions are available to them? What is the information of the players? What are the preferences of the players over the possible outcomes of play? Hence, one is forced to think in detail about the economic institutions and instruments that could play a role in a certain context, a particular phenomenon one wants to explain.

When transforming an economic situation into a game, one faces the trade-off between simplicity and abstraction on the one hand, and reality on the other. A game should be reasonably simple in order to be analyzed and to provide insights, but it should also not be too far from reality. One needs to take care in defining the rules of the game. Sometimes a small change of the rules of the game is sufficient to lead to totally different results. Consider, for example, a very simple linear duopoly model with two identical firms. When the firms compete in prices profits will be zero in equilibrium (Bertrand (1883)). Equilibrium profits are positive and identical (Cournot (1838)) if they compete in quantities. If one firm can move before the other, the leader will have higher profits than the follower (Von Stackelberg (1934)). This example shows that the exact specification of the rules may have an enormous impact on the outcome. Sometimes it is difficult to determine which details of a real life problem are truly irrelevant for the decision problem, and which only appear to be so at first sight.

It is therefore important to examine the consequences when players can make, what Thomas Schelling (1960) calls, strategic moves. Examples of such moves are: making promises, uttering threats or committing oneself to a certain course of action. The notion of commitment should be interpreted in a broad sense. One way of committing oneself is to take an irrevocable action. For instance, an army unit may burn the bridges behind itself, thereby making a withdrawal physically impossible. Another way of committing is to delegate responsibility of the final decision to another person whose incentive structure provides an ex post motive for fulfillment. For example, a government may give authority to punish lawbreakers to sadists. Yet another way to commit is to worsen one's own payoff in case of non-fulfillment. For example, one may accept a wager not to break a diet. These and many other examples of strategic moves are given by Schelling (1960) in his classic *The strategy of conflict*. He shows (without having to go into deep technical detail) that such moves may affect the outcome. He also points out that, in order to make commitments or promises, it is necessary that they can be communicated to the other players.

A major part of this thesis is devoted to the analysis of games in which strategic moves are possible. These strategic moves should and will be formally incorporated in the description of the rules of the game.

In Chapters 4 and 5 we consider a strategic move that appears to be completely irrelevant. Namely, we consider a situation in which one of two players involved in a particular game may choose to burn some money (or, more general, some utility) before this game is played. At first sight it seems obvious that this player will never choose to burn money, and that the possibility to do so will have no effect on the outcome of the game. This intuition, however, is misleading.[1] The reason is that once the possibility of burning money is incorporated in the rules of the game, the second player must make his choice of action dependent on whether or not some money was burnt. The model with burning money is therefore mathematically equivalent to a model in which the first player has access to a costly communication technology. The outcome of the game will depend on how a signal of burning money (or, for that matter, a signal of not burning money) is interpreted by the second player. In Chapter 4 a signal may reveal intended play in the future, while in Chapter 5 a signal may reveal some private information.

Chapters 6, 7 and 8 of this thesis deal with the order in which decisions are made. Usually it is assumed either that all players choose simultaneously (nobody can make commitments), or that they choose in a particular given order (the first player can (or must) make a commitment). Anyway, the order of moves is exogenous and this may be a bad reflection of reality. Some authors addressed this issue by determining the outcome for every possible order of moves and then compare the different outcomes. This approach is, however, unsatisfactory. Usually it will not be the case that all players prefer the outcome that results from one particular order and it will be difficult to say something sensible about which order will occur. It does not necessarily make sense to assume that, if no agreement of the above type can be reached, then the model in which all players play simultaneously will result. It might be the case, like in a price-setting duopoly with differentiated products, that a leader-follower configuration is preferred, by leader and follower, to the configuration with simultaneous moves. And even in case there is one particular order of moves that is preferred by all players, it is not at all obvious that this order will result.

[1] See also Van Damme (1989) and Ben-Porath and Dekel (1992).

In order to be able to predict whether there will be sequential or simultaneous decision making, one needs a model in which the order of the moves is determined endogenously. In Chapters 6 and 7 we use a two period model. In this model players decide whether to move early (period 1) or late (period 2). However, they do not have the ability to move *before* the other. Chapter 6 deals with games in which players have finitely many strategies. The main result concerns games that have an equilibrium at which no player has an incentive to move first. Chapter 7 is concerned with games like Cournot competition, in which players have infinitely many strategies and do have an incentive to move first. In Chapter 8 we investigate the robustness of the power to commit with respect to the communication of the commitment. Schelling (1960) already pointed out that the communication of a commitment is crucial. Moreover, Bagwell (1992) claimed that the power to commit is completely lost whenever there is the slightest imperfection in the observability of the commitment. In the case of quantity competition this would imply that, if there is some small probability that the follower observes a quantity different from the actual commitment quantity, the leader will not commit himself to his Stackelberg quantity. Instead, he will commit himself to his Cournot quantity. We show, however, that Bagwell's claim crucially depends on the restriction to pure strategy equilibria, and that, when mixed strategy equilibria are taken into account, there is no need to reconsider the literature that applies the idea of a first-mover advantage.

..., and Solutions

Once the rules of a game are established, mathematical and game theoretical tools can be fruitfully applied to "solve" the game analytically and rigorously. A solution of a game should either predict how this game will be played or recommend a particular way of playing the game. Usually a game is solved by determining the Nash equilibria, that is, by finding those strategy profiles from which no player wants to deviate unilaterally. Sometimes refinements or "coarsenings" of the Nash equilibrium concept are considered. In the last two decades the game theoretical tool-box has been extended by many authors. The main contributions, however, were by John Nash, John Harsanyi and Reinhard Selten.[2] Nash (1950) introduced the concept of a (Nash) equilibrium. Selten (1965) introduced the notion of subgame perfection in dynamic games to exclude

[2]For their contributions they were awarded the 1994 Nobel memorial prize in economics by the Royal Swedish Academy of Sciences.

Nash equilibria that are based on empty threats. Harsanyi (1967-68) showed how situations with incomplete information can be modeled. Moreover, Harsanyi and Selten (1988) addressed the problem that a game may possess multiple equilibria. Together they developed a theory that always selects a unique solution.

By now there are many different solution concepts, many of which are refinements of the Nash equilibrium. (See Van Damme (1987).) The solution of a game depends crucially on the solution concept that is employed. This is particularly (but not exclusively) true if the game has more than one Nash equilibrium. Should all equilibria be considered as solutions of the game? Or should perhaps attention be restricted to a certain refinement of the Nash equilibrium concept? And if so, to which one? Should we invoke the selection theory of Harsanyi and Selten (1988) which gives a unique solution, or is it more reasonable not to restrict to Nash equilibria but to consider all rationalizable strategies (Bernheim (1984) and Pearce (1984))? There seems to be no definitive answer to these questions, and this thesis certainly will not give one. In this thesis we do not introduce new solution concepts. Instead, we discuss and apply several solution concepts that have been introduced only recently and which may therefore be unfamiliar to most readers. We also consider some variations of existing solution concepts.

Main emphasis will be on *curb sets*. A curb set is a product set of strategies that is closed under best replies. If a player is sure that all other players will play a strategy from a particular curb set, then it is in this player's interest to do so as well. Curb sets were introduced by Basu and Weibull (1991). In Chapter 2 we formally introduce curb sets and other related set-valued concepts. In Chapter 3 we show how players may learn to play strategies from a minimal curb set. In Chapters 4, 5 and 6 (and, in a sense, also in Chapters 7 and 8) we employ minimal curb sets as the solution concept.

In Chapters 7 and 8 we will use an equilibrium selection theory that selects a unique strategy profile as the solution of the game. This theory is mainly based on the selection theory of Harsanyi and Selten (1988), and in particular on the tracing procedure of Harsanyi (1975). In some respects, however, our theory will differ from the existing theories. Most applications of the selection theory of Harsanyi and Selten (1988) have been restricted to the class of 2×2 games. Probably this is due to the complexities of the computations involved in their theory. Chapters 7 and 8 may therefore also be seen as exercises in applying the tracing procedure.

1.1 Summary

Games, Rules, and Solutions is a thesis on non-cooperative game theory. It aims to investigate the consequences of incorporating certain strategic moves (such as commitments and communication) in the rules of a game. At the same time it explores the workings of some recently introduced solution concepts.

Chapter 1 is introductory.

Chapter 2 starts with a brief discussion of (set-valued) solution concepts. Several set-valued concepts are formally defined in this chapter. Main focus will be on sets of strategy profiles that are closed under (some kind of) best replies. In particular, curb, curb*, robust and persistent sets and primitive, primitive*, robust and persistent formations are defined. Several properties of these concepts are listed. We will make use of these properties in several other chapters. Examples are provided to illustrate the subtle differences between the above mentioned solution concepts. Chapter 2 concludes by discussing two other set-valued solution concepts, namely cyclically stable sets (Matsui (1992)) and equilibrium evolutionarily stable sets (Swinkels (1992)).

It is important to know whether a solution concept is relevant in an eductive or in an evolutionary context. Chapter 3 shows that curb and persistency are relevant in an evolutionary context. It presents a dynamic learning process with the following characteristic: Players have a bounded memory and play best replies against beliefs, formed on the basis of strategies used in the recent past. It is shown that this learning process leads the players to playing strategies from a minimal curb set, i.e. a minimal set of strategies that is closed under best replies. This result continues to hold in the presence of mimickers and sophisticated players. When players are not certain about the strategies chosen by the other players, the process does not converge to a minimal curb set, but to a minimal curb*, robust or persistent set, depending on how the uncertainty is modeled.

Chapters 4 and 5 examine the consequences of allowing some players to burn some money before a game is played. At first sight it might seem that an opportunity to burn money is completely irrelevant, since it seems unlikely that it will ever be used. It turns out, however, that having this opportunity is (mathematically) equivalent with having access to a costly pre-play communication technology and that it does affect the outcome. In Chapter 4 we consider a setting with an arbitrary (finite) number of players among which there are some who have the possibility to burn money. We show that strategy profiles in minimal curb (or curb*) sets yield all players who have the possibility to burn, their preferred outcome. Moreover, in such profiles no money is actually burnt, the possibility alone suffices. The results go through for persistent sets in the special case of two person games, but not for games with more than two players. Chapter 5 considers the possibility of burning money in a game with asymmetric information. There are two players, a Sender and a Receiver. The Sender has private information on his type, which affects the payoff of both players. The Receiver must choose an action, and this action affects the payoff of both players. Before the Receiver takes an action the Sender sends a costly message to the Receiver. It is shown that each type of the Sender gets his preferred action in any curb (or curb* or persistent) strategy profile.

It has been long recognized that the order in which players take their decisions is very important. In most models the rules of the game prescribe a particular order in which players make their decisions. Sometimes there is a "first-mover advantage". If that is the case all players will try to obtain the first move. That is, each player will want to commit himself to a particular action, thereby inducing the other players to behave in a way that is favorable to himself.

Chapter 6 investigates which equilibria of a bi-matrix game are still viable when both players have the opportunity to commit themselves. To that end we study a model of endogenous timing in which players face the trade-off between committing early and forcing the opponent to best respond, and moving late so as to be able to play a best response against the opponent. It is shown that when the sequencing of the moves is endogenous, mixed strategy equilibria of the original game are only viable if they are commitment robust, that is, if no player has an incentive to move first at this equilibrium. In contrast, any pure strategy equilibrium is a perfect equilibrium outcome of the timing game. The concepts of curb* and persistent equilibria, however, allow the conclusion that only commitment robust equilibria are viable.

Chapter 7 analyzes some economic games of timing, where the original game has a unique Nash equilibrium which is not commitment robust. The endogenous timing game has three pure subgame perfect equilibria. Either a leader-follower configuration occurs in which the players play in different periods (two possibilities) or the equilibrium of the original game is played in period one. In order to make a selection between these equilibria elements from the equilibrium selection theory of Harsanyi and Selten (1988) are applied. In particular, the risk dominance relation between the three equilibria is investigated. It turns out that playing the Nash equilibrium in period one is risk dominated by both Stackelberg equilibria. In order to select between the Stackelberg equilibria we restrict ourselves to three specific games: (1) Cournot quantity competition,

(2) price competition with differentiated products and (3) the private provision of a public good. We assume that players differ with respect to marginal cost. The result we obtain in each of the three games under consideration, is that the equilibrium in which the low cost firm is the leader is risk dominant.

In Chapters 6 and 7 it was assumed that players can commit to an irreversible action, and communicate this action to the opponent. In Chapter 8 we analyze the situation with an exogenously given leader. But the action of this leader is only imperfectly observed by the follower. Bagwell (1992) made the claim that the power to commit oneself to an action does not confer any strategic benefit if this commitment cannot be perfectly observed by the opponent. It is shown in Chapter 8 that this claim crucially depends on the restriction to pure strategy equilibria. Specifically, the game analyzed by Bagwell always has a mixed equilibrium that is close to the Stackelberg equilibrium of the game in which the commitment is observed perfectly. We introduce a new theory of equilibrium selection that combines elements from the theory of Harsanyi and Selten (1988) with elements from the theory of Harsanyi (1993). When the noise is sufficiently small, this theory selects the Stackelberg equilibrium.

1.2 Organization of this thesis

Several chapters of this thesis are based on papers that have appeared elsewhere. Some of these were co-authored by Eric van Damme. Chapter 3 is based on Hurkens (1994). Chapter 4 is almost the same as Hurkens (1993). Chapter 6 is a (substantially) revised version of Van Damme and Hurkens (1993). Chapter 7 is based on joint research with Eric van Damme. Chapter 8 is almost identical to Van Damme and Hurkens (1994).

This thesis can be read in several ways. The first way is to start at the beginning and read through till the end. Alternative ways may be followed by readers who are mainly interested in a particular chapter or subject. All chapters are almost self-contained.

Readers interested in the theoretical aspects of set-valued solution concepts may read Chapter 2 (for definitions, properties and some examples) and Chapter 3 (for a justification of such concepts in a dynamic learning framework). They are urged to read Section 1.3 (that introduces some notation and basic concepts) first. Readers mainly interested in applications of set-valued solution concepts should glance through Sections 1.3 and 2.2 and then go to Chapters 4, 5 or 6. Persons interested in equilibrium selection theories may want to read Chapters 7 or 8. Readers interested in Chapter 7 may want to read Sections 6.1 through 6.3 first, while Chapter 8 may be read without any preparatory reading.

By organizing the thesis in such a way that each chapter is (almost) self-contained, it was inevitable that some overlap between chapters occurred. So be it.

1.3 Preliminaries

This section introduces some basic notations and concepts that will be used throughout most parts of the thesis.

Let $g = (A_1, \ldots, A_n, u_1, \ldots, u_n)$ be a finite n-person normal form game with player set $N = \{1, \ldots, n\}$. A_i denotes the finite set of pure strategies of player i. Let $A = \prod_{i=1}^n A_i$. For any finite and non-empty set X we denote the set of probability distributions over X by $\Delta(X)$. Let $S_i = \Delta(A_i)$ denote the set of mixed strategies of player i, and let $S = \prod_{i=1}^n S_i$. We will identify a pure action with the probability distribution that puts all weight on this action. For a strategy profile $s \in S$, s_i denotes player i's strategy and s_{-i} denotes the profile of strategies of all players besides i. Similarly, for any product set $X = \prod_{i=1}^n X_i$ we denote $X_{-i} = \prod_{j \neq i} X_j$. Usually we will interpret S_i and S_{-i} as beliefs over the behavior of player i and the behavior of all players besides i, respectively. Sometimes we will allow a player i to have a correlated belief over the behavior of all other players, i.e. his belief will then be an element of $\Delta(A_{-i})$. The uncorrelated strategy sets S and S_{-i} are embedded in a natural way in $\Delta(A)$ and $\Delta(A_{-i})$, respectively. Typical elements of these correlated strategy sets are denoted by s^c and s^c_{-i}, respectively. For a distribution $s^c \in \Delta(A)$ let $s^c_i \in \Delta(A_i)$ be the marginal on A_i, and let $s^c_{-i} \in \Delta(A_{-i})$ be the marginal on A_{-i}, i.e.

$$\begin{aligned} s^c_i(a_i) &= \textstyle\sum_{a_{-i} \in A_{-i}} s^c(a_i, a_{-i}) && (a_i \in A_i) \\ s_{-i}(a_{-i}) &= \textstyle\sum_{a_i \in A_i} s^c(a_i, a_{-i}) && (a_{-i} \in A_{-i}) \end{aligned}$$

Because of the natural embedding of mixed strategies into the set of correlated strategies it suffices to give definitions for the most general case, i.e. for the case of correlated strategies. The payoff function of player i, $u_i : A \to \mathbf{R}$, is extended to $u_i : S_i \times \Delta(A_{-i}) \to \mathbf{R}$ in the following natural way.

$$u_i(s_i, s^c_{-i}) := \sum_{a \in A} s_i(a_i) s^c_{-i}(a_{-i}) u_i(a)$$

For $s^c_{-i} \in \Delta(A_{-i})$ let $\mathcal{B}_i(s^c_{-i}) = \arg\max\{u_i(s_i, s^c_{-i}) | s_i \in S_i\}$ denote the set of best replies for player i against s^c_{-i}. For $s \in S$ let $\mathcal{B}(s) = \prod_{i=1}^n \mathcal{B}_i(s_{-i})$. The pure best replies are denoted by $B_i(s^c_{-i}) = \mathcal{B}_i(s^c_{-i}) \cap A_i$ and $B(s) = \mathcal{B}(s) \cap A$. For any set $F \subset \Delta(A)$, let

$B_i(F) = \cup_{s^c \in F} B_i(s^c_{-i})$, and let $B(F) = \prod_{i=1}^n B_i(F)$. A strategy combination s is called a Nash equilibrium if $s \in \mathcal{B}(s)$. It is called strict if $\{s\} = \mathcal{B}(s)$. As a kind of inverse of the best reply correspondence we define the stability region. Now we need to distinguish the correlated strategy case from the uncorrelated strategy case. For $s_i \in S_i$, let

$$\begin{aligned} \mathrm{St}_i(s_i) &= \{s_{-i} \in S_{-i} | s_i \in \mathcal{B}_i(s_{-i})\}, \\ \mathrm{St}^c_i(s_i) &= \{s^c_{-i} \in S^c_{-i} | s_i \in \mathcal{B}_i(s^c_{-i})\}. \end{aligned}$$

Strategies s_i and s'_i are equivalent ($s_i \sim s'_i$) if $u_i(s_i, a_{-i}) = u_i(s'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$. Strategy s_i is weakly dominated by s'_i if $u_i(s_i, a_{-i}) \leq u_i(s'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ with strict inequality for at least one a_{-i}. For $s_i \in S_i$ we let $C(s_i) = \{a_i \in A_i | s_i(a_i) > 0\}$ denote the carrier or support of s_i, and $C(s) = \prod_{i=1}^n C(s_i)$. Sometimes, we also write $\mathrm{supp}(s) = C(s)$. Strategy s_i is completely mixed if $C(s_i) = A_i$. Strategy combination s is completely mixed if $C(s) = A$.

Chapter 2

Set-valued Solution Concepts

2.1 Introduction

Since the work of Nash (1950), the Nash equilibrium is probably the best known and most applied solution concept in the literature on strategic interaction. Actually, it was already present (under different names) in the work of Cournot (1838) and Bertrand (1883). A Nash equilibrium describes a situation of "stability". It is a profile of strategies such that no player can improve his payoff by a unilateral deviation. Any theory that would recommend players to play a strategy profile that is not a Nash equilibrium, is necessarily self-defeating. There are, however, some problems with the Nash equilibrium concept.

First, it is optimal to play an equilibrium strategy if the other players play their part of the equilibrium. But, in general, a Bayesian player should try and guess what strategies the other players are choosing and respond optimally against these strategies. It is, a priori, not obvious that the other players choose equilibrium strategies.[1] The literature has paid attention to the question of how players may reach a Nash equilibrium. In some circumstances, players can deduce what the other players will do. When it is common knowledge that players are rational (i.e., maximize their expected payoff), iterated elimination of strictly dominated strategies can be applied, and in some situations (when the game is dominance solvable (see Moulin (1979) and Milgrom and Roberts (1991))) this is enough to ensure that a Nash equilibrium will be played. In other cases, a player may have some idea of which strategies will be used by the other players, because the

[1]See Edgar Allen Poe's (1908) "The purloined letter" for a nice story in which a boy beats his opponent in Matching Pennies by guessing his opponent's strategy and playing optimally against that strategy.

same game has been played repeatedly over time and he has some information about the evolution of play. There is a growing literature on learning and evolution, which views (some) Nash equilibria as the rest points of a dynamic system in which players adapt their strategies gradually and myopically. (For a survey see Van Damme (1994a).)

A second problem with Nash equilibria is that they might not be that stable after all. Although no player can improve his payoff by a unilateral deviation, it might be the case that some deviations do not hurt the player. In other words, some players might have multiple best replies. This is always the case in mixed strategy equilibria. This problem does not arise for strict equilibria. The conclusion seems to be that strict Nash equilibria are very appealing[2] but, unfortunately, in many interesting games they do not exist.

We remarked above that any *single-valued* solution concept, that is, any theory that recommends to play a particular strategy profile, must recommend a Nash equilibrium or be self-defeating. In this chapter we will consider *set-valued* solution concepts. Unlike a Nash equilibrium, such solution concepts do not give a single strategy profile as the solution of the game, but they yield a set of strategy profiles as the solution. Set-valued solution concepts are not new: Bernheim (1984) and Pearce (1984) proposed the notion of rationalizable strategies, Kalai and Samet (1984) defined persistent retracts, Gilboa and Matsui (1991) and Matsui (1992) introduced cyclically stable sets (CSS), Swinkels (1992) introduced equilibrium evolutionarily stable sets (EES sets). More abstract notions of set-valued solution concepts were proposed by Kalai and Schmeidler (1977) and Gilboa and Samet (1991). Some of these set-valued concepts can be viewed as a generalization of strict equilibria in the following sense. If a player deviates and plays a (sensible) strategy outside the solution set, then he is not playing optimally. Moreover, many of these set-valued solution concepts exist for all games.

One might wonder whether set-valued solution concepts are useful for prediction purposes. Probably (and hopefully) it will be more often the case that observed behavior (i.e., the strategy profile actually chosen) is an element of the solution set, than that it happens to be a Nash equilibrium. But if the set is large the predictive power may be very low. For example, the set may contain all strategy profiles. In Chapter 4 on pre-play communication we will see that the solution set (in that case curb or persistent retracts) is large, but all elements of this set correspond to the same outcome. In Chapter 6 on endogenous timing, we will see that all equilibria in a persistent retract correspond to

[2]One strict equilibrium may be more appealing than another because the former may be payoff and/or risk dominant.

the same outcome. Other applications of persistent and curb retracts can be found in Kalai and Samet (1985), Blume (1993a,b, 1994) and Balkenborg (1993a). We conclude that the assumption that players choose a strategy from the solution set is rather weak, but in some applications it suffices to draw strong conclusions.

It could be the case that set-valued solution concepts suffer from the same problem as Nash equilibria. Namely, it is only optimal to play in the set if the other players do so. The literature has paid attention to this matter also. The solution concepts EES and CSS are usually viewed as the resulting outcome of some (unspecified) evolutionary dynamic. Ritzberger and Weibull (1991) provide a similar dynamic where the "rest points" (or better, the "rest sets") are sets closely related to curb retracts. Chapter 3 presents an explicit learning model that leads players to play strategies from a curb (or persistent) retract.

The rest of this chapter is organized as follows. In Section 2.2 several set-valued solution concepts are defined and the main properties of and relations between these concepts are stated. Many examples are given to illustrate the subtle differences between these concepts. Section 2.3 contains the definitions of EES sets and CSS. We try to clarify the relations between these concepts (and curb retracts). We also correct some mistakes made in Swinkels (1992).

2.2 Defining properties

2.2.1 Independent beliefs

Let $g = (A, u)$ be a normal form game. In this section we will consider solution concepts that recommend to each player i a finite and non-empty subset $C_i \subset S_i$. The recommendation will be public, i.e. it is commonly known by the players that the recommendation is $C = \prod_{i=1}^{n} C_i$. If player i is Bayesian he will form some beliefs about the behavior of the other players. If we assume that a player believes that the other players will follow the recommendation, and if we furthermore assume independence, then the belief of player i is represented by an element of the set $\prod_{j \neq i} \Delta(C_j)$.[3] Following Kalai and Samet (1984) we call $R = \prod_{i=1}^{n} \Delta(C_i)$ a retract. Hence, a retract is a product of convex hulls of finitely many (mixed) strategies. We call R a selection retract if each element of C_i is equivalent with a pure strategy and no two strategies in C_i are equivalent. We are interested in self-enforcing solution concepts and therefore we will insist that these retracts are closed

[3] See Section 2.2.2 for the case where a player's belief is dependent, i.e. is an element of $\Delta(\prod_{j \neq i} C_j)$.

under best replies: If $s_{-i} \in R_{-i}$ and $s_i \in \mathcal{B}_i(s_{-i}) \setminus R_i$ then player i might very well play a strategy outside R_i. If other players recognize this possibility, they may assign positive probability to the event that s_i is played, that is, they may believe that player i does not follow the recommendation. Since not all best replies are equally compelling (some may be weakly dominated, inadmissible or not semi-robust (these concepts are defined below)), we will consider several different requirements.[4] The whole set of strategies is always closed under any kind of best replies, but for prediction purposes (or as a recommendation) it is not very useful. Therefore we will impose that the solution be minimal. A better motivation for the minimality condition is provided by Chapter 3. Let us now define the different kinds of best replies.

Definition 2.1 **(i)** *s_i is an undominated best reply to s_{-i} (denoted by $s_i \in \mathcal{UB}_i(s_{-i})$) if it is a best reply to s_{-i}, and s_i is not weakly dominated by any $s_i' \in S_i$.*

(ii) *s_i is an admissible best reply to s_{-i} (denoted by $s_i \in \mathcal{AB}_i(s_{-i})$) if there exists a sequence $\{s_{-i}^{(k)}\}_{k=1}^{\infty}$ of completely mixed strategies in S_{-i} that converges to s_{-i}, such that s_i is a best reply to each element of this sequence.*

(iii) *s_i is a semi-robust best reply to s_{-i} (denoted by $s_i \in \mathcal{SRB}_i(s_{-i})$) if there exists a sequence $\{s_{-i}^{(k)}\}_{k=1}^{\infty}$ in the interior (in the topological space S_{-i}) of the stability region of s_i that converges to s_{-i}.*

We will use these notions of best replies to define several properties. Then we will consider minimal sets with these properties.

Definition 2.2 *Let $R = \prod_{i=1}^{n} R_i$ be a retract.*

(i) *R is closed under best replies:*

$$\text{for all } i \in N \text{ and for all } s_{-i} \in R_{-i}, \;\; \mathcal{B}_i(s_{-i}) \subset R_i$$

(ii) *R is closed under undominated best replies:*

$$\text{for all } i \in N \text{ and for all } s_{-i} \in R_{-i}, \;\; \mathcal{UB}_i(s_{-i}) \subset R_i$$

(iii) *R is closed under admissible best replies:*

$$\text{for all } i \in N \text{ and for all } s_{-i} \in R_{-i}, \;\; \mathcal{AB}_i(s_{-i}) \subset R_i$$

[4]If one of these requirements would be obviously more compelling than the others, we could restrict attention to the most compelling requirement. This is however not the case.

(iv) *R is closed under semi-robust best replies:*

$$\text{for all } i \in N \text{ and for all } s_{-i} \in R_{-i},\;\; \mathcal{SRB}_i(s_{-i}) \subset R_i$$

(v) *R is closed under semi-robust best replies up to equivalence:*

$$\text{for all } i \in N \text{ and for all } s_{-i} \in R_{-i} \text{ and } s_i \in \mathcal{SRB}_i(s_{-i}),$$

$$\text{there exists } s_i' \in R_i \text{ with } s_i' \sim s_i$$

(vi) *R is absorbing:*

$$\text{for all } i \in N \text{ there exists an open neighborhood } \mathcal{O}_{-i} \subset S_{-i} \text{ of } R_{-i}$$

$$\text{such that for all } s_{-i} \in \mathcal{O}_{-i} \text{ there exists } s_i \in R_i \text{ with } s_i \in \mathcal{B}_i(s_{-i})$$

(vii) *R satisfies the Nash requirement:*

$$\text{for all } s_{-i} \in R_{-i} \text{ there exists } s_i \in R_i \text{ with } s_i \in \mathcal{B}_i(s_{-i})$$

A retract is minimal with respect to some property (X) if it satisfies (X) and it does not contain a smaller retract that satisfies (X). Retracts that are minimal with respect to properties (i), (ii), (iii), (iv), (vi) and (vii) are called, respectively, curb, curb, admissible, robust, persistent and Nash retracts.*

The name "curb" was introduced by Basu and Weibull (1991) and is mnenomic for closed under rational behavior. Persistent retracts were first introduced by Kalai and Samet (1984). The other terminology is borrowed from Balkenborg (1992). It is easy to see that a set is a Nash retract if and only if it is a singleton containing a Nash equilibrium. Generic normal form games have no equivalent strategies and all best replies are semi-robust. Hence, for such games curb retracts and robust retracts coincide.

Lemma 2.1 *Let R be a retract. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Rightarrow$ (vii). Moreover, for a two person game we have (iii) $\Rightarrow$ (ii).*

Proof. Most of the implications are trivial. The claim on two person games follows from the fact that a strategy that is not weakly dominated must be an admissible best reply to some strategy of the opponent. See e.g. Pearce (1984, Appendix B, Lemma 4). We will only show (v) $\Leftrightarrow$ (vi).

First note that a mixed strategy is a semi-robust best reply only if it is the mixture of pure semi-robust best replies that are equivalent. Furthermore, two equivalent strategies have the same stability region.

Suppose that R is an absorbing retract and let $s_i \in \mathcal{SRB}_i(s_{-i})$ for some $s_{-i} \in R_{-i}$. Every open neighborhood of R_{-i} intersects the interior of $St_i(s_i)$. Up to equivalence player i has a unique best reply in the interior. Hence, there exists $s_i' \in R_i$ with $s_i' \sim s_i$. This establishes $(vi) \Rightarrow (v)$.

Now suppose that R is closed under semi-robust best replies up to equivalence. Let U_{-i} be the union of all stability regions of semi-robust best replies against R_{-i}, i.e.

$$U_{-i} = \cup_{\{s_i \in \mathcal{SRB}_i(s_{-i}) | s_{-i} \in R_{-i}\}} St_i(s_i).$$

It can easily be checked that the interior of U_{-i} is an open neighborhood of R_{-i} that satisfies the requirement in (vi). This completes the proof. □

We will now describe the structure of and the relation between the different retracts defined above. In Chapters 4, 5 and 6 we will make use of the properties listed in the next theorem.

Theorem 2.1 **(i)** *Every curb retract is the cartesian product of convex hulls of sets of pure strategies. Two different curb retracts have an empty intersection. The same assertions hold for curb*, admissible and robust retracts.*

(ii) *A persistent retract is a selection retract.*

(iii) *Every game has a curb retract. Every curb retract contains a curb* retract. Every curb* retract contains an admissible retract. Every admissible retract contains a robust retract. Every robust retract contains a persistent retract. Every persistent retract contains a Nash equilibrium.*

(iv) *If R is a curb retract and $s_i \in R_i$ is an extreme point of R_i, then there exists $s_{-i} \in R_{-i}$ such that $s_i \in \mathcal{B}_i(s_{-i})$. Similar assertions hold for curb*, admissible, robust and persistent retracts (with the appropriate notion of best reply).*

Proof. (i) If a mixed strategy is an extreme point of a curb retract then it is not a best reply to any strategy combination of the retract. (Otherwise all pure strategies in its carrier would be in the retract which would contradict the mixed strategy being an extreme point.) Hence, the retract that is obtained by deleting all strategies that put positive weight on this mixed strategy is also closed under best replies. But this

contradicts the fact that the retract is minimal. The other assertions are trivial: They follow basically from the observation that the intersection of two retracts that are closed under best replies, is closed under best replies.
(ii) This follows from Lemma 2.1. For a formal proof see Kalai and Samet (1984).
(iii) By (i) we know that curb retracts are spanned by pure strategies. Since there are only finitely many pure strategies and since the set of all strategy profiles is closed under best replies we have that there is at least one curb retract. The other assertions follow from Lemma 2.1.
(iv) This follows from the minimality argument used in part (i). □

Note that non-extreme points of a curb (or other) retract need not be best replies. Even stronger, a persistent retract may contain a mixed strategy that is strictly dominated. Since retracts are actually described by the sets of extreme points, we will introduce some terminology for these sets. Let us call a finite product set $C = \prod_{i=1}^{n} C_i \ (\subset S)$ a curb set if $\prod_{i=1}^{n} \Delta(C_i)$ is closed under best replies. It follows that a minimal curb set is the set of extreme points of a curb retract. Similarly, we define (minimal) curb*, admissible, robust, absorbing and persistent sets.

By Theorem 2.1(iii) we may use the set-valued solution concepts to refine the notion of a Nash equilibrium. For example, call a Nash equilibrium a curb equilibrium if it is contained in a curb retract. By the above theorem a curb equilibrium always exists. Note, however, a persistent equilibrium is not always a curb equilibrium and a curb equilibrium is not necessarily persistent. (We will see some examples later on.) This means that we cannot say that a curb equilibrium is a stronger (or weaker) concept than a persistent equilibrium. This is due to the minimality requirements for retracts. A retract that is closed under best replies is necessarily absorbing. However, a retract that is minimal with respect to closedness under best replies is not necessarily minimal with respect to absorbingness. There is a way of refining the Nash equilibrium concept in a "nested" way. Namely, it follows from Theorem 2.1 that every game has an equilibrium that is curb, curb*, admissible, (semi-)robust[5] and persistent. By Kalai and Samet's (1984) result that every persistent retract contains a proper (Myerson (1978)) equilibrium, we may even insist that this equilibrium is proper as well. This is clearly the strongest refinement we can get out of the set-valued solution concepts defined so far. Such a refinement, however,

[5]Here we use "semi-robust" to differentiate this equilibrium concept from Okada's (1983) concept of a robust equilibrium (which need not exist).

is not very appealing.[6] Why should an equilibrium that satisfies all these requirements be more attractive than any other proper (and persistent) equilibrium? We close this part with a remark. If a game has a unique persistent retract, and this retract is also closed under best replies, then every persistent equilibrium is curb, and vice versa. However, even in this case there may be persistent equilibria that are not proper and there may be proper equilibria that are not persistent. Examples of such games can be found in Van Damme (1987).

In the remainder of this section we give examples to illustrate the subtle differences between the solution concepts defined up to now. In particular, these examples show that a curb* equilibrium need not be curb, that an admissible equilibrium need not be curb*, etcetera.

A curb* retract is not necessarily contained in a curb retract.

Consider the game presented in Figure 2.1.

	L	C	R
T	1,1	0,1	0,0
B	0,1	1,0	1,2

Figure 2.1.

Only $\{(B,R)\}$ is a curb retract, but $\{(T,L)\}$ is a curb* retract.

An admissible retract is not necessarily contained in a curb* retract.

Consider the three person game presented in Figure 2.2. Each player i chooses between a_i and b_i.

a_3:

	a_2	b_2
a_1	1,1,0	0,0,1
b_1	0,0,1	0,0,0

b_3:

	a_2	b_2
a_1	0,0,0	0,0,0
b_1	0,0,0	1,1,1

Figure 2.2.

[6]Harsanyi (1993) defines an equilibrium to be eligible whenever it is both persistent and proper. He uses this notion for his new theory of equilibrium selection.

It is easy to check that a_3 is the unique admissible best reply (in *un*correlated strategies) against (a_1, a_2). It follows that $\{(a_1, a_2, a_3)\}$ is an admissible retract. But $\{(b_1, b_2, b_3)\}$ is the only curb* (or curb) retract.

A robust retract is not necessarily contained in an admissible retract.

Consider the game presented in Figure 2.3.

	a_2	b_2	c_2	d_2
a_1	0,2	2,0	0,1	0,0
b_1	2,0	0,2	0,1	0,0
c_1	1,0	1,0	0,0	0,1
d_1	0,0	0,0	1,0	1,1

Figure 2.3.

This is a two person game without dominated strategies. Hence, the admissible retracts coincide with the curb retracts. The only curb retract is $\{(d_1, d_2)\}$. However, $R_1 \times R_2$ is a robust retract when $R_i = \Delta(\{a_i, b_i\})$.

A persistent retract is not necessarily contained in a robust retract.

Consider the game presented in Figure 2.4.

	L	C	R
T	2,2	0,2	0,0
B	0,0	1,0	1,1

Figure 2.4.

The only robust retract is $\{(B, R)\}$. There are many persistent retracts. Let $s(\alpha) = (1-\alpha)L + \alpha C$. For all $\alpha \in [0, 2/3)$ we have that $\{(T, s(\alpha))\}$ is a persistent retract. This example also shows that a Nash retract need not be contained in a persistent retract, since $\{(T, s(2/3))\}$ is such a retract.

2.2.2 Dependent beliefs

Thus far we assumed that if $\prod_{i=1}^{n} C_i$ is recommended, then player i's belief is represented by an element of $R_{-i} = \prod_{j \neq i} \Delta(C_j)$. That is, we assumed that beliefs are independent.

However, it might be the case that player i believes that the actions of other players are correlated. He might *believe* this, even in the case the other players do in fact not correlate their actions. Such a belief is represented by an element of $\Delta(\prod_{j\neq i} C_j)$. Of course, R_{-i} is in a natural way embedded in the latter set, hence a player may still have independent beliefs. In this section we will examine the consequences of allowing for dependent beliefs. Again, we will consider sets that are closed under best replies, but now more beliefs are allowed and, as a consequence, more strategies will have to be included in the recommendation. We will also consider undominated, admissible and semi-robust best replies. Some of these concepts need to be adapted to the correlated strategy case. Before we do that let us remark that it is obvious that for two person games nothing will change.

The notions of best replies and undominated best replies need not be adapted: Weakly domination is checked against all pure strategy combinations.

We say that s_i is an admissible best reply in correlated strategies against $s^c_{-i} \in \Delta(A_{-i})$ (denoted by $s_i \in \mathcal{AB}^c_i(s^c_{-i})$) if there exists a sequence $\{s^{c,(k)}_{-i}\}_{k=1}^{\infty}$ of completely mixed correlated strategy combinations that converges to s^c_{-i}, such that s_i is a best reply to each element of this sequence. Note that if s^c_{-i} is in fact an uncorrelated strategy combination, then $\mathcal{AB}_i(s^c_{-i}) \subset \mathcal{AB}^c_i(s^c_{-i})$. In general, the latter inclusion is strict.

We say that s_i is a semi-robust best reply in correlated strategies against s^c_{-i} (denoted by $s_i \in \mathcal{SRB}^c_i(s^c_{-i})$) if there exists a sequence $\{s^{c,(k)}_{-i}\}_{k=1}^{\infty}$ in the interior (in the topological space $\Delta(A_{-i})$) of the stability region $St^c_i(s_i)$ that converges to s^c_{-i}. Again, if s^c_{-i} is in fact an uncorrelated strategy combination , then $\mathcal{SRB}_i(s^c_{-i}) \subset \mathcal{SRB}^c_i(s^c_{-i})$, and this inclusion is in general strict.

Again, we will consider minimal sets that are closed under different kinds of best replies (against dependent beliefs). For convenience we will work directly with the set of extreme points. Following Harsanyi and Selten (1988) we will use the terminology "formation" instead of sets or retracts, to distinguish the correlated strategy case from the uncorrelated strategy case.

Definition 2.3 *Let $C = \prod_{i=1}^{n} C_i$, where $C_i \subset S_i$ is finite and non-empty.*

(i) *C is a primitive formation if it is minimal w.r.t.*

$$\text{for all } i \in N \text{ and for all } s^c_{-i} \in \Delta(C_{-i}), \mathcal{B}_i(s^c_{-i}) \subset \Delta(C_i)$$

(ii) *C is a primitive* formation if it is minimal w.r.t.*

$$\text{for all } i \in N \text{ and for all } s^c_{-i} \in \Delta(C_{-i}), \mathcal{UB}_i(s^c_{-i}) \subset \Delta(C_i)$$

(iii) *C is an admissible formation if it is minimal w.r.t.*

$$\text{for all } i \in N \text{ and for all } s^c_{-i} \in \Delta(C_{-i}), \mathcal{AB}^c_i(s^c_{-i}) \subset \Delta(C_i)$$

(iv) *C is a robust formation if it is minimal w.r.t.*

$$\text{for all } i \in N \text{ and for all } s^c_{-i} \in \Delta(C_{-i}), \mathcal{SRB}^c_i(s^c_{-i}) \subset \Delta(C_i)$$

(v) *C is a persistent formation if it is minimal w.r.t.*

$$\text{for all } i \in N \text{ there exists an open neighborhood } \mathcal{O}^c_{-i} \subset \Delta(A_{-i}) \text{ of } \Delta(C_{-i}),$$

$$\text{such that for all } s^c_{-i} \in \mathcal{O}^c_{-i} \text{ there exists } s_i \in \Delta(C_i) \text{ with } s_i \in \mathcal{B}_i(s^c_{-i})$$

Analogies of Lemma 2.1 and Theorem 2.1 can be shown easily. There is, however, a slight difference with the last part of Lemma 2.1. There we stated that, for games with two players, curb* retracts and admissible retracts coincide. In the world of correlated strategies a player views his opponents as one single player. It is easy to adjust the proof in Pearce (1984, Appendix B) to prove that, for all $s^c_{-i} \in \Delta(A_{-i})$, $\mathcal{AB}^c_i(s^c_{-i}) = \mathcal{UB}_i(s^c_{-i})$. It follows that, for all games, admissible formations and primitive* formations coincide.

It is easy to see that every persistent formation contains a persistent set: Let C be a persistent formation and let $\mathcal{O}^c_{-i}$ be an open neighborhood of $\Delta(C_{-i})$ with the required property. Now let $\mathcal{O}_{-i} = \mathcal{O}^c_{-i} \cap S_{-i}$ denote the set of uncorrelated strategies in this open neighborhood. Then $\mathcal{O}_{-i}$ is an open neighborhood of $\prod_{j \neq i} \Delta(C_j)$ and it has the required property.

The following example of a three person game shows that it matters whether one allows dependent beliefs or not.

A minimal curb set is not necessarily contained in a primitive formation.

Consider the three person game in Figure 2.5, which is due to Balkenborg (1992).

a_3:

	a_2	b_2	c_2
a_1	1,1,4	1,2,4	0,0,0
b_1	2,1,4	2,2,1	0,0,0
c_1	0,0,0	0,0,0	0,0,0

b_3:

	a_2	b_2	c_2
a_1	2,2,1	2,1,4	0,0,0
b_1	1,2,4	1,1,4	0,0,0
c_1	0,0,0	0,0,0	0,0,0

c_3:

	a_2	b_2	c_2
a_1	0,0,3	0,0,0	0,0,0
b_1	0,0,0	0,0,3	0,0,0
c_1	0,0,0	0,0,0	5,5,5

Figure 2.5.

The strategy (c_1, c_2, c_3) is a strict equilibrium and defines the unique primitive formation (and, for that matter, also the unique primitive*, robust and persistent formation). The product set $\{a_1, b_1\} \times \{a_2, b_2\} \times \{c_1, c_2\}$ is a minimal curb set. Note, however, that c_3 is the best reply against the correlated strategy $1/2(a_1, a_2) + 1/2(b_1, b_2)$. From this example it also follows that minimal curb*, robust and persistent sets are not necessarily contained in primitive*, robust and persistent formations, respectively.

In Figure 2.6 we summarize the relations between the different concepts encountered in this chapter. Here $X \supset Y$ means that every X contains a Y, and that, in general, not every Y is contained in an X. We already provided an example for each case "$X \supset Y$" that shows that not every Y is contained in an X. Whenever the inclusion symbol is accompanied by an asterisk ($X \supset^* Y$), then every Y is an X for the special case of two person games.

$$\begin{array}{ccccccccc}
\text{prim. form.} & \supset & \text{prim.* form.} & = & \text{adm. form.} & \supset & \text{robust form.} & \supset & \text{pers. form.} \\
\cup_* & & \cup_* & & \cup_* & & \cup_* & & \cup_* \\
\text{min. curb set} & \supset & \text{min. curb* set} & \supset^* & \text{min. adm. set} & \supset & \text{min. robust set} & \supset & \text{pers. set}
\end{array}$$

Figure 2.6.

2.3 CSS and EES

The final section of this chapter is devoted to the discussion of two other set-valued solution concepts, namely CSS and EES. In contrast to the earlier introduced solution concepts, these concepts will not be important in the remainder of this thesis. Skipping this section will therefore not lead to any difficulties in any of the other chapters.

Swinkels (1992) introduced the notion of an equilibrium evolutionarily stable set (EES set). An EES set is a closed and non-empty set of Nash equilibria that is stable under a dynamic evolutionary process. The evolutionary process is not modeled explicitly. The stability requirement is described by an entry condition for mutants. Roughly speaking, this entry condition says that a small portion of mutants can enter (and 'survive') if the strategy they play is a best response to the strategy of the post-entry population. We will refer to this entry condition as (S) (for Swinkels). Gilboa and Matsui (1991) introduced the notion of a cyclically stable set (CSS). Matsui (1992) modified the notion of a CSS. This modified version of CSS (mCSS) is a set of strategy combinations that is closed under any dynamic that always moves in the direction of a best reply. The dynamic implicitly present in Matsui's concept is therefore a best reply dynamic, and not an evolutionary dynamic. Although the intuitive motivations for EES and mCSS are quite different, the two concepts are, in a mathematical sense, closely related. Matsui showed that an mCSS is a closed and non-empty set of strategy profiles that satisfies a certain entry condition. We will refer to this entry condition as (M) (for Matsui). The different entry conditions (S) and (M) look similar, and it is often stated informally that the only difference between CSS and EES is the extra requirement for EES that it should be a subset of the set of Nash equilibria.[7] In fact, Swinkels claims that any set satisfying (S) also satisfies (M). What Swinkels in fact "shows" is that any set that satisfies (S) also satisfies a condition (T) ((T) for "two"). However, his proof is only correct for two person games. Moreover, condition (T) is only equivalent to (M) in the case of two person games.

In this section we will restore the mistakes made by Swinkels. We will show an example of a game that admits a non-empty set of Nash equilibria that satisfies (M) but not (S). Hence, this example illuminates the fact that the closedness condition is more important than Swinkels admits. Then we show by example that in a three person game, an EES set need not satisfy (T). Then we will give a correct proof for Swinkels' claim that any closed set satisfying (S) also satisfies (M). We show that even generic normal form games need not have EES sets. Then we state some results on the relations between EES sets, CSS and curb retracts. We conclude this section by showing that there is no link between EES sets and forward induction.

2.3.1 Defining CSS and EES

Throughout this section we will only speak about uncorrelated strategies. In particular, we view $s \in S$ as an L-tuple, where $L = \sum_{i=1}^{n} |A_i|$, and not as a distribution from $\Delta(A)$. Hence, for any strategies s and s' and any number $\alpha \in [0,1]$, we will write $\alpha s + (1-\alpha)s'$ to denote the strategy combination in which each player i plays $\alpha s_i + (1-\alpha)s'_i$. We will use the notation $u_i(s; s'_i)$ to denote i's payoff when he plays s'_i while each player $j \neq i$ plays s_j.

We will only employ Matsui's modified version of CSS.

[7] Blume, Kim and Sobel (1993) define entry resistant sets as minimal closed and non-empty sets satisfying (S). They state that they do not know whether these ER sets are the same as mCSS.

Definition 2.4 (Matsui (1992)) *Strategy profile $s' \in S$ is* directly accessible *from $s \in S$ if there exist $T \geq 0$, a continuous function $p : [0,T] \to S$, differentiable from the right, and a step function $h : [0,T) \to S$, continuous from the right, such that*
(i) $p(0) = s$, $p(T) = s'$,
(ii) $\frac{d^+p}{dt}(t) = h(t) - p(t)$ (for all $t \in [0,T)$) and
(iii) $h(t) \in \mathcal{B}(p(t))$ (for all $t \in [0,T)$).

Hence, strategy profile s' is directly accessible from s if there exists a well-behaved function $p : [0,T] \to S$ such that the curve it describes starts at s, ends at s' and 'moves' at any point $t \in [0,T)$ in the direction of a best reply to $p(t)$. Matsui calls this function $p(\cdot)$ a best response dynamic path.

Now accessibility is defined recursively as follows: s' is accessible from s if either (i) s' is directly accessible from s, or (ii) there exists a sequence $\{s'_k\}_{k=1}^{\infty}$ converging to s' such that s'_k is accessible from s for each k, or (iii) s' is accessible from some $\hat{s}$, which in turn is accessible from s.

Definition 2.5 *A non-empty set $\Theta \subset S$ is called a modified cyclically stable set (mCSS) if, for all $s \in \Theta$ and all $s' \in S$ it holds that s' is accessible from s if and only if $s' \in \Theta$.*

Matsui shows that an mCSS always exists, using Zorn's Lemma. In order to apply Zorn's Lemma, he needs that $R(s) = \{s' \in S | s' \text{ is accessible from } s\}$ is closed. This follows of course from (ii) in the recursive definition. It is, however, not clear whether the same result could not be obtained from direct accessibility alone.

We need one more definition. For given $\Theta \subset S$ and $s \in \Theta$, a vector $w \in \mathbb{R}^{\sum_{i\in N}|A_i|}$ is a *feasible direction* from s in Θ if there exists $\bar{\varepsilon} > 0$ such that $s + \varepsilon w \in \Theta$ for all $\varepsilon \in [0, \bar{\varepsilon}]$. Now we are ready to state Matsui's characterization.

Lemma 2.2 (Matsui (1992)) *A subset $\Theta \subset S$ is an mCSS if and only if it is minimal with respect to*

(1) *Θ is closed and non-empty, and*

(M) *If $s \in \Theta$ and $w \in \mathbb{R}^{\sum_{i\in N}|A_i|}$ with $s + w \in S$, and $\varepsilon' > 0$ is such that for all $\varepsilon \in [0, \varepsilon')$, for all $i \in N$ and for all $z_i \in \mathcal{B}_i(s)$*

$$u_i(s + \varepsilon w; s_i + w_i) \geq u_i(s + \varepsilon w; z_i),$$

then w is a feasible direction from s in Θ.

Proof. See Matsui (1992). □

It will be more convenient to work with another characterization of an mCSS.

Lemma 2.3 *$\Theta \subset S$ is an mCSS if and only if it is minimal with respect to*

(1) *Θ is closed and non-empty, and*

(M*) *If $s \in \Theta$ and $s' \in S$ and $\tilde{\varepsilon} > 0$ is such that for all $\varepsilon \in [0, \tilde{\varepsilon})$,*

$$s' \in \mathcal{B}((1-\varepsilon)s + \varepsilon s'),$$

then $(1-\varepsilon)s + \varepsilon s' \in \Theta$ for all $\varepsilon \in [0, \tilde{\varepsilon})$.

Proof. We show that Θ satisfies (1) and (M) if and only if it satisfies (1) and (M*). It is obvious that Θ satisfies (M) whenever it satisfies (M*). So, let Θ satisfy (1) and (M) and let $s \in \Theta$, $s' \in S$ and $\tilde{\varepsilon} > 0$ be such that

$$u_i((1-\varepsilon)s + \varepsilon s'; s_i') \geq u_i((1-\varepsilon)s + \varepsilon s'; a_i)$$

for all $\varepsilon \in [0, \tilde{\varepsilon})$, for all $i \in N$ and for all $a_i \in A_i$. By (M) there exists $\varepsilon' > 0$ such that $(1-\varepsilon)s + \varepsilon s' \in \Theta$ for all $\varepsilon \in [0, \varepsilon']$. Let $\bar{\varepsilon}$ be the supremum over all ε' with this property. Since Θ is closed the supremum is in fact a maximum. We need to show that $\bar{\varepsilon} \geq \tilde{\varepsilon}$. Suppose that $\bar{\varepsilon} < \tilde{\varepsilon}$. We will derive a contradiction.

Consider the strategy profile $\bar{s} = (1-\bar{\varepsilon})s + \bar{\varepsilon}s' \in \Theta$. For all $\lambda \in [0, \tilde{\varepsilon} - \bar{\varepsilon})$ we have $(1-\lambda)\bar{s} + \lambda s' = (1 - ((1-\lambda)\bar{\varepsilon} + \lambda))s + ((1-\lambda)\bar{\varepsilon} + \lambda)s'$, while $(1-\lambda)\bar{\varepsilon} + \lambda < \bar{\varepsilon} + \lambda < \tilde{\varepsilon}$. Hence,

$$u_i((1-\lambda)\bar{s} + \lambda s'; s_i') \geq u_i((1-\lambda)\bar{s} + \lambda s'; a_i)$$

for all $a_i \in A_i$ and all $i \in N$. Hence, $s' - \bar{s}$ is a feasible direction from $\bar{s}$ in Θ: There exists $\varepsilon' > 0$ such that $(1-\varepsilon)\bar{s} + \varepsilon s' \in \Theta$ for all $\varepsilon \in [0, \varepsilon']$. However, this contradicts the fact that $\bar{\varepsilon}$ is the maximal ε' such that $(1-\varepsilon)s + \varepsilon s' \in \Theta$ for all $\varepsilon \in [0, \varepsilon']$, since $(1-\varepsilon')\bar{s} + \varepsilon' s' = (1-\varepsilon')(1-\bar{\varepsilon})s + ((1-\varepsilon')\bar{\varepsilon} + \varepsilon')s'$, and $(1-\varepsilon')\bar{\varepsilon} + \varepsilon' > \bar{\varepsilon}$. □

Now we recall the definition of an EES set from Swinkels (1992).

Definition 2.6 *$\Theta \subset S$ is an equilibrium evolutionarily stable set (EES set) if it is minimal with respect to*

(1) *Θ is closed and non-empty,*

(2) *$\Theta \subset \{s \in S | s \in \mathcal{B}(s)\}$, and*

(S) *there exists $\delta' > 0$ such that for all $\delta \in (0, \delta')$, for all $s \in \Theta$ and for all $s' \in S$*

$$s' \in \mathcal{B}((1-\delta)s + \delta s') \quad \Rightarrow \quad (1-\delta)s + \delta s' \in \Theta.$$

The entry conditions (M*) and (S) are very similar. The difference between them is that the $\tilde{\varepsilon}$ in condition (M*) is chosen for each pair of strategy combinations $(s, s') \in \Theta \times S$ separately, while the δ' in condition (S) has to be chosen uniformly. Often it is stated that the difference between CSS and EES is the additional requirement for the latter to be a set of Nash equilibria. The following example shows that the conditions (M*) and (S) are different. It also shows that the closedness condition is important. Consider the game from Figure 2.7.

	L	C	R
T	2,2	0,2	0,0
B	0,0	1,0	1,1

Figure 2.7.

Let $\Theta = \{(T, (1-\alpha)L + \alpha C) | \alpha \in [0, 2/3)\}$. This set contains only equilibria and it satisfies (M*) (and (M)): For each $s(\alpha) := (T, (1-\alpha)L + \alpha C) \in \Theta$ we may choose $\tilde{\varepsilon} = 2/3 - \alpha$. Θ does, however, not satisfy (S): For each $\delta' \in (0, 1]$ we can take $\delta = \delta'/2$, $s' = (B, C)$ and $s = s(\alpha)$ with α such that $(1-\alpha)(1-\delta) = 1/3$. Then we have that $\tilde{s} := (1-\delta)s + \delta s' = ((1-\delta)T + \delta B, 1/3L + 2/3C)$ and $s' \in \mathcal{B}((1-\delta)s + \delta s')$. Condition (S) is now violated since $\tilde{s} \notin \Theta$. Of course, in our example Θ is not closed. It is not known whether any closed set of Nash equilibria that satisfies (M*) necessarily satisfies (S).

Swinkels already showed by example that an EES set need not exist. However, his example was non-generic. One might conjecture that for generic (two person) normal form games always at least one EES set exists. This is not true. We will give an example below. First we will discuss the alternative condition formulated by Swinkels.

(T) For all $s \in \Theta$ and for all $s' \in S$ that are Nash equilibria in the game restricted to $\mathcal{B}(s)$, there exists $\delta' > 0$ such that $(1-\delta)s + \delta s' \in \Theta$ for all $\delta \in [0, \delta')$.

Swinkels introduced this condition because it is easier to verify than (S). Moreover, he claimed that (T) is the entry condition used in Matsui's characterization of an mCSS, and that (T) is implied by (S). It is not difficult to check that, for two person games, (T) is equivalent with (M*). But this is no longer true for games with more than two players. Consider the three player game in Figure 2.8. Each player i chooses between a_i and b_i.

	a_2	b_2
a_1	3,3,3	1,3,1
b_1	3,1,1	0,0,0

a_3

	a_2	b_2
a_1	1,1,3	0,0,0
b_1	0,0,0	0,0,0

b_3

Figure 2.8.

The set of Nash equilibria of this game consists of two components. The first component just contains b. The second component is $X_1 \cup X_2 \cup X_3$, where X_i is the set of all strategy profiles in which each player $j \neq i$ plays a_j for sure. Let Θ be the second component, that is, Θ is the set of all Nash equilibria except b. It is not difficult to check that Θ is an EES set and an mCSS. But Θ does not satisfy (T): b is a Nash equilibrium in the (restricted) game where players choose strategies in $\mathcal{B}(a)$. If Θ satisfies (T), then there exists $\varepsilon > 0$ such that $(1-\varepsilon)a + \varepsilon b \in \Theta$, but that strategy profile is clearly not an equilibrium.

Swinkels claimed that any EES set must contain an mCSS. What he in fact "showed" was that any EES set satisfies (T). His proof is only correct for two person games. We will now prove Swinkels' original claim.

Lemma 2.4 *If Θ satisfies (1) and (S) then it also satisfies (M).*

Proof. Let Θ be closed and let $\delta' > 0$ satisfy the requirement in (S). We will show that Θ satisfies (M*). Suppose not.

Let $s \in \Theta$, $s' \in S$ and $\tilde{\varepsilon} > 0$ be such that for all $\varepsilon \in [0, \tilde{\varepsilon})$, all $i \in N$ and all $a_i \in A_i$

$$u_i((1-\varepsilon)s + \varepsilon s'; s_i') \geq u_i((1-\varepsilon)s + \varepsilon s'; a_i), \tag{2.3.1}$$

but suppose that $(1-\varepsilon)s + \varepsilon s' \notin \Theta$ for some $\varepsilon \in [0, \tilde{\varepsilon}]$. Let $\bar{\varepsilon} = \sup\{\varepsilon' \in [0, \tilde{\varepsilon}] | (1-\varepsilon)s + \varepsilon s' \in \Theta \text{ for all } \varepsilon \in [0, \varepsilon']\}$. It follows that $0 \leq \bar{\varepsilon} < \tilde{\varepsilon}$. Consider the strategy profile $\bar{s} = (1-\bar{\varepsilon})s + \bar{\varepsilon}s' \in \Theta$. By (2.3.1) and (S) it follows that for all $0 < \varepsilon < \min\{\tilde{\varepsilon} - \bar{\varepsilon}, \delta'\}$, $(1-(\varepsilon+(1-\varepsilon)\bar{\varepsilon}))s + (\varepsilon+(1-\varepsilon)\bar{\varepsilon}))s' = (1-\varepsilon)\bar{s} + \varepsilon s' \in \Theta$. This contradicts the definition of $\bar{\varepsilon}$. □

Now we will give a generic example of a two person game that does not admit an EES set. Note that it follows from our previous observations that in two person games any EES set satisfies (T). In the example we will show that there does not exist a non-empty set of Nash equilibria that satisfies (T). It follows then that there does not exist an EES set.

	α	β	γ	δ
a	6,6	2,2	0,7	2,0
b	2,2	6,6	2,0	0,7
c	7,0	0,2	1,1	0,0
d	0,2	7,0	0,0	1,1

Figure 2.9.

The game from Figure 2.9 is not really generic in the sense that the same numbers enter the payoff matrix several times. Moreover, the game is symmetric. However, the following arguments hold for all small perturbations of the payoffs. First note that pure strategy equilibria do not exist. Second, $\frac{1}{2}c+\frac{1}{2}d$ is strictly dominated by $\frac{1}{2}a+\frac{1}{2}b$, hence c and d cannot both belong to the carrier of an equilibrium. Straightforward calculations show that the only possibility for an equilibrium s with two pure strategies in its carrier is when $C(s) = \{a, b\} \times \{\alpha, \beta\}$. There are also two equilibria with three pure strategies in the carrier. Without perturbations the equilibria are $s^1 = ((\frac{1}{2}, \frac{1}{2}, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0, 0))$, $s^2 = ((\frac{4}{9}, \frac{3}{9}, \frac{2}{9}, 0), (\frac{4}{9}, \frac{3}{9}, \frac{2}{9}, 0))$ and $s^3 = ((\frac{3}{9}, \frac{4}{9}, 0, \frac{2}{9}), (\frac{3}{9}, \frac{4}{9}, 0, \frac{2}{9}))$. Any EES set must be made up of some of these equilibria. However, no such set satisfies (T): If s^2 is contained in the set, s^1 may enter since it is an equilibrium of the game restricted to $\mathcal{B}(s^2)$. This implies that, for small $\varepsilon > 0$, $(1-\varepsilon)s^2 + \varepsilon s^1$ is contained in the set, but this profile is not an equilibrium. The same story applies when s^3 is assumed to be contained in the set. Hence, only $\{s^1\}$ remains as a candidate for an EES set. However, (a, α) is an equilibrium in the game restricted to $\mathcal{B}(s^1)$, hence, this set does not satisfy (T).

The following theorem lists some relations between EES, mCSS and curb retracts.

Theorem 2.2 *Every EES set contains an mCSS. Every curb retract contains an mCSS. If R is a curb retract and $u(s) = \bar{u}$ for all $s \in R$, then R is an EES set and an mCSS. In the case of two person games a singleton mCSS is also an EES set.*

Proof. Most assertions are trivial so we only prove the last one. We already observed that in two person games (M*) is equivalent to (T). Hence, if $\{s^*\}$ is an mCSS then s^* must be an equilibrium, and it is the unique equilibrium of the game restricted to $\mathcal{B}(s^*)$. Let m^+ and m^- denote the maximum and minimum payoff that a player can obtain in the game, respectively. Choose $\delta' > 0$ such that, for all i and all $a_i \notin B_i(s^*)$

$$(1-\delta')u_i(s^*) + \delta' m^- > (1-\delta')u_i(s^*; a_i) + \delta' m^+.$$

It follows that for all $\delta \in [0, \delta')$ and all $s \in S$, $B((1-\delta)s^* + \delta s) \subset B(s^*)$. In particular, if $s \in \mathcal{B}((1-\delta)s^* + \delta s)$, then s is, under all best replies against s^*, a best reply against itself. But this means that s is an equilibrium in the game restricted to $\mathcal{B}(s^*)$. By presupposition this only happens in case $s = s^*$. Hence, δ' does the job in condition (S). □

2.3.2 Forward induction

Let $g = (A, u)$ be a two person normal form game. Suppose that this game has an equilibrium s^*, such that $u_1(s^*) > x > u_1(s)$, for some (fixed) number x and for all equilibria $s \neq s^*$. Let g^{out} denote the game obtained from g by giving player 1 an outside option, yielding him a payoff of x (and yielding player 2 an arbitrary number). Hence, in g^{out} player 1 first decides whether to enter the subgame g or to take his outside option. The notion of forward induction (Van Damme (1989)) then says that entering the subgame is a signal, indicating that player 1 intends to play s_1^*. Player 2 should recognize this signal and play s_2^*, his part of the equilibrium. If he indeed does so, player 1 will enter the subgame and his preferred equilibrium will be played.

When new solution concepts are introduced their usefulness is often demonstrated by an example of an outside option game in which the solution predicts the outcome that is consistent with forward induction. Especially solution concepts with an evolutionary flavor seem to be in favor of forward induction. (See Nöldeke and Samuelson (1993).) Matsui (1992) showed that mCSS captures the notion of forward induction when the underlying game is a battle of the sexes (or something slightly more general). Swinkels (1992) noted that an EES set need not contain an element consistent with forward induction. However, the example he gives is not correct.[8] His example can be easily corrected but then the underlying game will be non-generic. Swinkels states that it is an open problem whether EES sets capture the notion of forward induction in generic (two person) games. We will resolve this issue now.

First note that in the outside option game there are two equilibrium outcomes. There is one equilibrium consistent with forward induction, and there are many equilibria in which the outside option is chosen for sure. By Swinkels (1992, Theorem 3) we know that an EES set is a maximal connected set of equilibria. This implies that there are two candidates for an EES set in the outside option game. With some abuse of notation we

[8]In his example there is a continuum of equilibria and there does not exist a number x such that $u_1(s^*) > x > u_1(s)$, for all equilibria s different from the preferred equilibrium s^*.

let s^* denote the preferred equilibrium of g, as well as the forward induction equilibrium of g^{out}. It is straightforward to show that $\{s^*\}$ is an EES set of g^{out} if and only if it is an EES set of g. But this implies that forward induction has very little to do with EES sets. Moreover, we have already seen that a generic game need not admit an EES set. If such a game is extended with an outside option there will be no EES set that contains the forward induction equilibrium. It may also happen that the underlying game does admit an EES set but that the preferred equilibrium does not form an EES set. Of course, also in this case the outside option game does not have an EES set that contains the forward induction equilibrium. Consider the game in Figure 2.10.

	L	C	R
T	25,4	20,6	68,2
M	33,17	32,12	48,20
B	17,5	16,8	80,4

Figure 2.10.

This game has three equilibria, namely $s^1 = ((\frac{1}{3},\frac{1}{3},\frac{1}{3}),(\frac{1}{3},\frac{1}{3},\frac{1}{3}))$, yielding $(\frac{113}{3},\frac{26}{3})$, $s^2 = ((\frac{3}{5},\frac{2}{5},0),(\frac{5}{7},0,\frac{2}{7}))$ yielding $(\frac{261}{7},\frac{46}{5})$ and $s^3 = ((0,\frac{1}{3},\frac{2}{3}),(0,\frac{2}{3},\frac{1}{3}))$ yielding $(\frac{112}{3},\frac{28}{3})$. Since $\frac{113}{3} > \frac{112}{3} > \frac{261}{7}$, player 1 prefers s^1. However, it is easy to check that $\{s^1\}$ is not an EES set, while $\{s^3\}$ is an EES set. The reader may check that the game with an outside option of $x \in (\frac{112}{3},\frac{113}{3})$ does not admit an EES set.

On the other hand, it may also happen that the preferred equilibrium does form an EES set, but that the outside option game has another EES set (in which the outside option is chosen) as well. Hence, even in this case EES sets are not particularly in favor of forward induction. See for example the game in Figure 2.11.

	α	β	γ	δ
a	0,4	15,0	0,15	0,0
b	0,4	0,0	15,0	0,15
c	0,4	0,15	0,0	15,0
d	1,1	4,0	4,0	4,0
out	3,3	3,3	3,3	3,3

Figure 2.11.

The mixed equilibrium that yields an expected payoff of (5,5) forms an EES set (both in the underlying 4×4 game and in the outside option game). However, the set of all Nash equilibria of the entire game that yield the expected payoff (3,3) is an EES set and it contains no elements consistent with forward induction.

Chapter 3

Learning by Forgetful Players

3.1 Introduction

A product set of pure strategies is said to be closed under best replies if all best replies against all possible mixtures of these strategies are contained in the set. Minimal sets with this property are called minimal curb sets (Basu and Weibull (1991)). As we will see later (and as readers of Chapter 2 already know), curb sets are closely related to the better known persistent retracts. Kalai and Samet (1984) showed that every game has at least one persistent retract and that every persistent retract contains at least one (proper) Nash equilibrium. This enabled them to introduce the persistent equilibrium as a refinement of the Nash equilibrium concept.

Both concepts have been used in the literature. Kalai and Samet (1985) used persistency to achieve efficiency in unanimity games that are repeated as long as no agreement is reached. Blume (1993a) used the persistent retract as a set-valued solution concept in sender receiver games. Blume (1993b) shows that, in one-sided cheap talk games, equilibria in minimal curb sets sometimes select the sender's preferred outcome. In Chapter 4 we show that, in games where several players have the possibility to send costly messages, minimal curb sets always select the outcome preferred by all senders. In Chapter 6 the concepts of curb and persistency are applied in games of endogenous timing. Balkenborg (1993a) used these concepts in finitely repeated games.

In most of these papers it is argued informally that the concepts of curb and persistency have a dynamic and evolutionary flavor. However, few or no attempts have been made to support this idea with an evolutionary foundation of the concepts.

We construct a dynamic learning process to support these concepts. Roughly speaking, the learning evolves as follows: A particular game is played at discrete points in time. For each role in this game there is a pool of players. At the beginning of each period one player is drawn from each pool. These players will play the game in that period. Players have a bounded memory. On the basis of strategies played in the recent past, they form expectations about the strategies the other players will use, and best respond to these expectations. We assume that different players within the same pool may have different beliefs and therefore they may choose different actions. It is shown that, if the memory is long enough, play will settle down in a minimal curb set.

In some respects our results are stronger than those obtained thus far in the literature on learning. First, in contrast to Young (1993) we do not need to restrict attention to a special class of games. Second, the set of curb strategies is a subset of the set of rationalizable strategies (Bernheim (1984) and Pearce (1984)). Hence, our learning process reduces the number of "plausible" strategies. This is in contrast with Milgrom and Roberts (1991) who show that a sequence that is consistent with adaptive learning will eventually lie within the set of serially undominated strategies, which is a superset of the set of rationalizable strategies. In the final section we show that it is the forgetfulness of the players that accounts for this difference.

From the main and basic theorem we derive several results for learning processes where players learn in a somewhat different way. Play still settles down in minimal curb sets when some players do not play best responses to past play, but are more sophisticated than that, or, on the contrary, are less sophisticated. If we allow players to have beliefs as if the other players in the game correlate their actions, play settles down in a primitive formation (Harsanyi and Selten (1988)), a variant of a minimal curb set. When players are uncertain, the process does not converge to a curb set but to related solution concepts as curb*, robust or persistent sets, depending on how the uncertainty is modeled. The learning processes presented in this chapter may give the reader some insight in the differences and similarities between these related concepts. We also characterize two classes of games where our results go through, even if the players only observe the outcomes of past play, instead of the strategies.

The rest of the chapter is organized as follows. In Section 3.2 we introduce some preliminaries concerning Markov chains and curb sets. Section 3.3 describes the model of learning as a Markov chain. Section 3.4 contains the main result: the ergodic sets of the Markov chain correspond one-to-one to the minimal curb sets of the underlying game. In Sections 3.5 and 3.6 the above mentioned variations of the learning process are considered. In Section 3.7 we consider the possibility that players make mistakes with small probability. Section 3.8 compares this chapter to Milgrom and Roberts (1991).

3.2 Preliminaries

Notation will be as introduced in Section 1.3. For readers who skipped Chapter 2, we introduce the definition and some properties of minimal curb sets. For other readers, we recall them.

Definition 3.1 *A non-empty cartesian product set $C = \prod_{i=1}^{n} C_i \subset A$ is said to be closed under best replies (or C is a curb set) if $B(\prod_{i=1}^{n} \Delta(C_i)) \subset C$. Such a set is called a minimal curb set if it does not properly contain a curb set. Strategies contained in minimal curb sets are called curb strategies.*

It is straightforward to show that $B(\prod_{i=1}^{n} \Delta(C_i)) = C$ for any minimal curb set C. The notion of curb sets was introduced by Basu and Weibull (1991). Curb is mnemonic for closed under rational behavior.

A strict Nash equilibrium is a curb set as a singleton. Strict Nash equilibria have almost all desired properties one can hope for, except existence. A lot of these properties carry over to minimal curb sets. For instance, every curb set contains the support of a proper equilibrium (Kalai and Samet (1984), Balkenborg (1992)). Moreover, every game has at least one minimal curb set since A is curb.

Minimal curb sets can be viewed as a set-valued generalization of strict equilibria: When an outsider recommends to all players to play strategies from a minimal curb set C, then all players will follow this recommendation if they expect the other players to do so. The comparison with strict equilibria is not completely valid: minimal curb sets may contain weakly dominated strategies.

Before we go further let us consider some examples where minimal curb sets have some cutting power.

	L	R
T	4,4	1,1
B	1,1	2,2

Figure 3.1a.

	L	R
O	3,3	3,3
T	4,4	1,1
B	1,1	2,2

Figure 3.1b.

	LL	LR	RL	RR
m^0T	4,4	4,4	1,1	1,1
m^0B	1,1	1,1	2,2	2,2
m^1T	3,4	0,1	3,4	0,1
m^1B	0,1	1,2	0,1	1,2

Figure 3.1c.

Example A. Let g be given by the normal form in Figure 3.1a. This is a pure coordination game. Since (T, L) and (B, R) are strict equilibria it is easy to see that $\{(T, L)\}$ and $\{(B, R)\}$ are minimal curb sets, and that there are no other ones. In particular, the support of the mixed equilibrium is not contained in any minimal curb set.

Example B. Now consider the game in which player 1 has the choice between playing the game from Figure 3.1a and an outside option O, yielding both players a payoff of 3. The normal form representation of this game is given in Figure 3.1b. This game has a unique minimal curb set, namely $\{(T, L)\}$.

These two examples are nice because the minimal curb sets are singletons, and hence consist of one strict Nash equilibrium. In the following example, in contrast to those above, the unique minimal curb set is not a singleton.

Example C. Suppose that player 1 can send one of two messages, m^0 or m^1, to player 2 before the game from Figure 3.1a is played. Suppose that it costs player 1 i units to send m^i. Let ma denote player 1's strategy "I send message m and choose action a" and let a^0a^1 denote player 2's strategy "I choose action a^i if I receive message m^i". Then the (reduced) normal form of the game with pre-play communication is given in Figure 3.1c. Now it can be checked that $\{m^0T\} \times \{LL, LR\}$ is the unique minimal curb set of this extended game. The set is not a singleton but it consists only of equilibria that involve sending the cheapest message and then playing the equilibrium preferred by player 1. In Chapter 4 similar results will be obtained for a whole class of games with n players among which k have the possibility to send a costly message.

In the next section we will describe the learning process by means of a Markov chain. Therefore we will need some basic notions from the theory of Markov chains.

A finite stationary Markov chain is characterized by a pair (X, P), where X is a finite state space and $P : X \times X \to [0, 1]$ is a transition matrix. The interpretation is that $P(x, x')$ is the probability that the process will move from x to x' in one period. We will denote $x \rightsquigarrow x'$ if there exist $k \in \mathbb{N} \cup \{0\}, x_0, \ldots, x_k \in X$ with $x_0 = x$, $x_k = x'$ and $P(x_i, x_{i+1}) > 0$ $(i = 0, \ldots, k-1)$. Now $\rightsquigarrow$ defines a weak order on X. Hence, we can define an equivalence relation on X:

$$x \sim y \quad \Leftrightarrow \quad x \rightsquigarrow y \text{ and } y \rightsquigarrow x$$

Let $[x]$ denote the equivalence class that contains x and let $Q = \{[x] | x \in X\}$ denote the set of equivalence classes. We define a partial order $\preceq$ on Q.

$$[x] \preceq [y] \quad \Leftrightarrow \quad y \rightsquigarrow x$$

The minimal elements with respect to the order $\preceq$ are called *ergodic sets*. The other elements are called *transient sets*. If the process leaves a transient set it can never return to that set. And if the process is in an ergodic set it can never leave this set. The elements of these sets are called *ergodic* and *transient states*, respectively. We have the following theorem.

Theorem 3.1 *In any finite Markov chain, no matter where the process starts, the probability after k steps that the process is in an ergodic state tends to 1 as k tends to infinity.*

Proof. See e.g. Kemeny and Snell (1976). □

3.3 The learning process

According to the Bayesian approach, a player forms some expectation about the strategies that will be played by the other players, and best responds to his expectation. How these expectations are formed is not clear. When the same game has been played before, possibly by different people, it seems reasonable to suggest that expectations are formed on the basis of information on past play. One way of using this information is to assume that a player's belief corresponds to the empirical frequency of strategies used in the past. This way of forming beliefs, known as fictitious play (Brown (1951) and Robinson (1951)), makes perhaps sense in matching models, but it is certainly not the only possible way of forming beliefs. One drawback of fictitious play is that it assumes that all people always form expectations in the same way. This implies that if different people have the same information, they will form the same beliefs and consequently they choose the same action. One can create some stochastic variability in the process by assuming that people only draw an incomplete sample of the information, as in Young (1993). There it is assumed that players learn how the game was played in m out of the most recent K periods. The players use a fictitious play rule to map samples into beliefs, and best respond to these beliefs. The great technical advantage of Young's approach is that the learning process can be described by a finite Markov chain on the state space $H = A^K$, consisting of all sequences of length K drawn from A. In order to determine the ergodic sets of such Markov chains, one needs only to specify which transitions occur with positive probability, and which occur with zero probability.

We will also describe a learning process by means of a finite Markov chain, but we allow more variability in the responses of the players. In fact, we allow the degree of variability that is present in Milgrom and Roberts' (1991) definition of adaptive play.[1]

[1]See Section 3.8 for a comparison between this chapter and Milgrom and Roberts (1991).

Let $g = (A, u)$ be an n-person normal form game. Fix a positive integer K. Suppose we have a finite population of individuals that is partitioned into non-empty classes $V_1, \ldots, V_n$. The members of V_i are candidates to play role i in the game, and they all have the same payoff function u_i. Let $t = 0, 1, 2, \ldots$ denote successive time periods. Game g is played once every period. In period t one individual is drawn from each class V_i. These individuals are going to play the appropriate roles in the game this period. We will refer to the individual that is drawn from V_i to play the game in the current period as player i, although the identity of this player may change from time to time. Player i receives some, but not necessarily all, information about play in the recent K periods. Then he chooses a pure strategy according to some rule. We will define below what kind of information a player may receive, and how he chooses a strategy as a function of this information. Then the players are put back in their class. This ends period t and we move up to period $t + 1$.

Since we will assume that all the rules are time-independent, this learning process can be described by a stationary Markov chain on the state space $H = A^K$. Call $\hat{h} \in H$ a *successor* of $h \in H$ if $\hat{h}$ is obtained from h by deleting the left most element and by adding some element $a \in A$ to the right. Let $r(\hat{h})$ denote the right most element of $\hat{h} \in H$. For $h = (a^{-K}, \ldots, a^{-1}) \in H$ let $\pi_i(h) = \{a_i^{-K}, \ldots, a_i^{-1}\}$ denote the set of strategies played by player i in the recent past. We will assume that our learning process is described by a transition matrix $P \in \mathcal{P}$, where $\mathcal{P}$ is defined as follows.[2]

Definition 3.2

Let $\mathcal{P}$ denote the set of transition matrices P, that satisfy for all histories $h, \hat{h} \in H$,

$$P(h, \hat{h}) > 0 \quad \Leftrightarrow \quad \begin{cases} \hat{h} \textit{ is a successor of } h, \textit{ and} \\ r_i(\hat{h}) \in B_i(s_{-i}) \textit{ for some } s_{-i} \in \prod_{j \neq i} \Delta(\pi_j(h)) \quad \textit{(all } i\textit{)} \end{cases}$$

We will give two interpretations of a learning process that is described by some $P \in \mathcal{P}$. The first interpretation is close to the model of Young (1993). Fix a positive integer L. Before player i chooses a strategy in period t, he receives information about how the game was played by player j in the recent past, for all $j \neq i$. He receives L draws <u>with</u> replacement from the set $\{a_j(t-K), \ldots, a_j(t-1)\}$. A way of thinking about this sampling procedure is that player i passively hears about L precedents concerning the way player j played the game before. But player i is unaware of the fact that he might hear about

[2] A transition matrix describes a learning process for a fixed game, g, and a fixed length of the memory, K. We will however suppress superscripts g and K.

the same precedent several times. Assume that all draws are independent, but more importantly, assume that each combination of draws occurs with positive probability. Player i's belief about the behavior of player j corresponds to the empirical frequency of strategies in the sample of size L. Hence, this belief is one of a finite number of possible probability distributions. Namely, let $h = (a(t-K), \dots, a(t-1))$ denote the recent history and let $\pi_j(h) = \{a_j(t-K), \dots, a_j(t-1)\}$ denote the set of strategies played by player j in the recent past. Now player i's belief about player j's behavior is contained in the set

$$Gr_j(h, L) = \{s_j \in \Delta(\pi_j(h)) | s_j(a_j) = l/L \text{ for some } l \in \{0, 1, \dots, L\}\}.$$

We call the set $Gr^i(h, L) = \prod_{j \neq i} Gr_j(h, L)$ the L-grid distribution space for i induced by h. Note that as L increases, the grid becomes finer and finer, and $Gr^i(h, L)$ "approaches" $\prod_{j \neq i} \Delta(\pi_j(h))$. There exists a 'generic' class of games for which it suffices, for the purpose of this chapter, to choose L sufficiently large. However, in general we need a little bit more and therefore we assume that our learning process is described by some $P \in \mathcal{P}$.

Another interpretation of a learning process that is described by a transition matrix $P \in \mathcal{P}$ is the following. Suppose that the individuals in a class have different personal characteristics: They use the information on past play to know which strategies will certainly not be used (namely the ones that have not been played in the recent history). But each individual makes his own personal assessment of the probabilities with which the remaining strategies will be played. Some people are very optimistic and expect the best, while others are very pessimistic and expect the worst. And there will be a lot who have some intermediate beliefs. Of course, we need sufficient diversity in the different classes when this learning process is to be described by some $P \in \mathcal{P}$. Note, however, that this does not necessarily mean that these classes are large. Suppose that for each strategy $a_i \in A_i$, there is some individual in V_i who plays a_i, whenever it is a best reply to some belief that puts positive weight only on strategies that were played recently. (And he chooses a best reply to the most recent strategy otherwise.) Then we only need $|A_i|$ individuals in class V_i.

In the next section we will state and prove the main theorem of this chapter: Play will settle down in a minimal curb set.

3.4 Ergodic sets

Fix $K \in \mathbb{N}$ as the length of the histories. Recall from Section 3.2 that $h \rightsquigarrow \hat{h}$ means that there exist $k \in \mathbb{N}$, $h^0, \ldots, h^k \in H = A^K$ such that $h^0 = h$, $h^k = \hat{h}$ and $P(h^i, h^{i+1}) > 0$. Now $\rightsquigarrow$ defines a weak order on H and hence we can define an equivalence relation on H and an order on the set of equivalence classes of H. We will be interested in the minimal elements of this order, the ergodic sets.

Let C be a minimal curb set of $g = (A, u)$. We say that $h \in H$ is a C-history if $h \in C^K$. We call h a *curb history* if it is a C-history for some minimal curb set C.

Now we are ready to state the main theorem.

Theorem 3.2 *There exists $\underline{K} \in \mathbb{N}$ such that for all $K \geq \underline{K}$ and every Markov chain with a transition matrix $P \in \mathcal{P}$*
(i) If $Z \subset H$ is an ergodic set then $Z \subset C^K$ for some minimal curb set C.
(ii) For every minimal curb set C there exists exactly one subset $Z \subset C^K$ that is ergodic.
(iii) For each minimal curb set C and each strategy $\bar{a} \in C$ there exists an ergodic state h with $r(h) = \bar{a}$.

The theorem states that, if the history is long enough, any ergodic set is a set of C-histories for some minimal curb set C and that the set of C-histories contains one ergodic set. Hence, the ergodic states are curb histories. Moreover, once the ergodic set contained in C^K is entered, every strategy $\bar{a} \in C$ is played infinitely often. From Theorem 3.1 then the following corollary follows.

Corollary 3.1 *The probability that the players are playing a curb strategy profile after k steps of the learning process tends to 1 as k tends to infinity, if histories are sufficiently long.*

The intuition for the theorem is quite clear. By having a large enough memory, players may have beliefs with large supports. This means that best replies against all kinds of mixtures will be played now and then. This creates so much stochastic variability that players sooner or later will play curb strategies. When they keep drawing the "right" samples, they will keep best responding against curb strategies, and hence they will play curb strategies again. It might happen that they will do this K periods in a row. The probability that this happens at a specific point in time is only small, but with probability one it will happen eventually. By that time all non-curb strategies will be forgotten. The strategies that will be played from that point on, will depend on the sample drawn, but it is sure that it will be curb strategies again.

Before we give a formal proof we make two remarks about Theorem 3.2. First, note that assertions (i) and (ii) do not imply that C^K is an ergodic set whenever C is a minimal curb set. Still, the reader may think that the only ergodic set contained in C^K is C^K itself. However, the following example shows that C^K need not be ergodic.

	a_2	b_2	c_2
a_1	4,1	1,4	2,3
b_1	1,4	4,1	2,3
c_1	3,2	3,2	0,0

Figure 3.2.

Consider the game in Figure 3.2. This game has only one curb set, namely the set of all pure strategy combinations. But the profile $\bar{h} = (c, c, \ldots, c)$ cannot be reached under the learning process from any other history. This is so because c is only a best reply against some mixtures of a and b. Hence, there exists no h with $P(h, \bar{h}) > 0$ and $\bar{h}$ is not contained in the ergodic set.

The second remark concerns the length of the histories. In the proof of Theorem 3.2 we will use a lower bound on K, but that bound is not tight. On the other hand, it is important that histories are not too short, as the example from Figure 3.3 shows.

	x	l	c	r
X	4,4	2,2	2,2	2,2
T	2,2	5,0	0,5	0,0
M	2,2	0,0	5,0	0,5
B	2,2	0,5	0,0	5,0

Figure 3.3.

It is not difficult to see that if $K = 2$, then the set of histories $\{(a^{-2}, a^{-1}) | a^{-j} \in \{T, M, B\} \times \{l, c, r\}\}$ contains an ergodic set. Take for example the history (Tl, Mr). Agents from pool V_1 will play a best reply against $\alpha l + (1-\alpha) r$, for some $\alpha \in [0, 1]$. Hence, they will play T or B. But the unique minimal curb set is the singleton $\{(X, x)\}$. So the history must not be too short. Note that if $K = 3$ and the process is in state (Tl, Mr, Mc), then there will be some agent in V_1 who will play X, since X is the best reply against $\frac{1}{3}l + \frac{1}{3}c + \frac{1}{3}r$.

Note that the game from Figure 3.3 has a unique equilibrium, namely (X, x). This equilibrium is strict. Since every curb set contains the support of a Nash equilibrium and since a strict equilibrium forms a curb set as a singleton, it follows that this game has a unique minimal curb set. Hence, if players behave as described by our learning process then they will eventually play the equilibrium. This reasoning holds for all games that have a unique equilibrium that happens to be strict. So we proved

Corollary 3.2 *Suppose that a is the unique Nash equilibrium of g and that a is strict. The probability that players are playing the equilibrium after k steps of the learning process tends to 1 as k tends to infinity, if histories are sufficiently long.*

The remainder of this section contains a formal proof of Theorem 3.2. First we introduce some notation and state a lemma.

Let F be a non-empty subset of A. We define the projection of F on A_i as $p_i(F) = \{f_i | f \in F\}$ and we define $\text{span}(F) = \prod_{i=1}^n p_i(F)$. Hence, $\text{span}(F)$ is the smallest cartesian product set in A that contains F. Similarly, for a history $h = (a^{-K}, \dots, a^{-1})$ we define $\pi_i(h) = \{a_i^{-K}, \dots, a_i^{-1}\}$ and $\text{span}(h) = \prod_{i=1}^n \pi_i(h)$. We say that $B \subset A$ spans F if $\text{span}(B) = \text{span}(F)$.

For a history h let $\mathcal{B}^{ind}(h) = \{s \in S | \text{supp}(s) \subset \text{span}(h)\}$. This set contains all independent beliefs a Bayesian player might have when the process is in state h. Similarly, we define for a set $F \subset A$, $\mathcal{B}^{ind}(F) = \{s \in S | \text{supp}(s) \subset \text{span}(F)\}$. Let $M = \max_i |A_i|$.

Lemma 3.1 *Let $a^1, \dots, a^T \in A$ be such that $a^{t+1} \notin span(\{a^1, \dots, a^t\})$ for all $t = 1, \dots, T-1$. Then $T \leq \sum_{i=1}^n |A_i| - (n-1)$.*

Proof. Easy and hence omitted. □

Proof of Theorem 3.2. Take $\underline{K} = \sum_{i=1}^n |A_i| - (n-1) + M$ and let $K \geq \underline{K}$. Let $P \in \mathcal{P}$.

Let $h^t = (x^{K-t}, \dots, x^1, a^1, \dots, a^t)$ be a particular history and assume that $F^t = \text{span}(\{a^1, \dots, a^t\})$ is not a curb set. Then there exists $a^{t+1} \in B(\mathcal{B}^{ind}(F^t)) \setminus F^t$. Let $h^{t+1} = (x^{K-t-1}, \dots, x^1, a^1, \dots, a^{t+1})$. Then $P(h^t, h^{t+1}) > 0$. Starting from an arbitrary history h^1 we can apply this argument repeatedly. By Lemma 3.1 we know that there exists $T \leq \underline{K} - M$ such that $h^1 \rightsquigarrow h^T = (x^{K-T}, \dots, x^1, a^1, \dots, a^T)$ and such that $F^T = \text{span}(\{a^1, \dots, a^T\})$ is a curb set. Let $C \subset F^T$ be a minimal curb set and let $\{b^1, \dots, b^M\}$ span C. Since every strategy in a minimal curb set is a best reply to some belief concentrated on this set and since $K \geq M + T$, we have $h^T \rightsquigarrow (\dots, a^1, \dots, a^T, b^1, \dots, b^M) \rightsquigarrow (b^1, \dots, b^M, b^M, \dots, b^M)$.

The above shows that for every history h, there exists a minimal curb set C such that for every set $\{b^1, \dots, b^M\}$ that spans C, we have $h \rightsquigarrow (b^1, \dots, b^M, b^M, \dots, b^M)$. Furthermore, the definition of $\mathcal{P}$ implies that if h is a C-history and $h \rightsquigarrow \hat{h}$, then $\hat{h}$ is also a C-history.

The second observation implies that the set of C-histories contains an ergodic set, for any minimal curb set C. The first observation then implies that the set of C-histories contains exactly one ergodic set, and that there are no other ergodic sets. Assertion (iii) follows from the observation that the spanning set $\{b^1, \ldots, b^M\}$ can be chosen such that $b^M = \bar{a}$. □

3.5 Variations on the same theme

We remarked before that one only needs to know which entries of the transition matrix are positive and which are zero in order to characterize the ergodic sets. In the proof of Theorem 3.2 we used that certain entries are positive (together with Lemma 3.1) to show that the process can move from any history h to a curb history $\hat{h}$ in a finite number of periods. Furthermore, we used the fact that certain entries are zero to ensure that the process can not leave the set of C-histories, for any curb set C.

It is possible to prove Theorem 3.2 for an even bigger class of transition matrices. Let $P \in \mathcal{P}$ and let $\tilde{P}$ be a transition matrix that satisfies, for any minimal curb set C,

$$P(h,\hat{h}) > 0 \;\Rightarrow\; \tilde{P}(h,\hat{h}) > 0 \tag{3.5.1}$$

$$h \in C^K \text{ and } \tilde{P}(h,\hat{h}) > 0 \;\Rightarrow\; \hat{h} \in C^K \tag{3.5.2}$$

Let $\tilde{\mathcal{P}}$ denote the set of all such transition matrices. It is obvious that Theorem 3.2 holds for all $P \in \tilde{\mathcal{P}}$. We will consider two subsets of $\tilde{\mathcal{P}}$, namely $\mathcal{P}^{soph}$ and $\mathcal{P}^{mim}$. The transition matrices in these sets correspond to learning processes where some players are more sophisticated (in the case of $\mathcal{P}^{soph}$) or less sophisticated (in the case of $\mathcal{P}^{mim}$). It turns out that for these two classes we can prove slightly stronger results.

3.5.1 More and less sophisticated players

Suppose that not all individuals in the classes are Bayesian players, but that some individuals are mimickers. Mimickers don't form expectations but just observe how other agents in the same role have played the game during (some of) the last K periods. Then they choose one of these strategies at random. When we retain our assumption about the Bayesian players, this learning process can be described by a transition matrix $P \in \mathcal{P}^{mim}$, where $\mathcal{P}^{mim}$ is defined as follows.

Definition 3.3
Let $\mathcal{P}^{mim}$ denote the set of transition matrices P, that satisfy for all histories $h, \hat{h} \in H$,

$$P(h,\hat{h}) > 0 \quad \Leftrightarrow \quad \begin{cases} \hat{h} \textit{ is a successor of } h, \textit{ and} \\ r_i(\hat{h}) \in B_i(\mathcal{B}^{ind}(h)) \textit{ or } r_i(\hat{h}) \in \pi_i(h) \quad (\textit{all } i) \end{cases}$$

Obviously, $\mathcal{P}^{mim} \subset \tilde{\mathcal{P}}$, hence Theorem 3.2 holds for all $P \in \mathcal{P}^{mim}$. We can prove a slightly stronger result: All curb histories are ergodic states.

Theorem 3.3 *There exists $\underline{K} \in \mathbb{N}$ such that for all $K \geq \underline{K}$ and for every Markov chain with a transition matrix $P \in \mathcal{P}^{mim}$, $Z \subset H$ is an ergodic set if and only if $Z = C^K$ for some minimal curb set C.*

Proof. Using the proof of Theorem 3.2, it suffices to show that if C is a minimal curb set and h and $\hat{h}$ are C-histories, then $h \rightsquigarrow \hat{h}$.

Let $\hat{h} = (a^{-K}, \ldots, a^{-1})$. We can choose a set $B = \{b^1, \ldots, b^M\}$ that spans C such that $a^{-j} \in \text{span}(\{b^{M-j+1}, \ldots, b^M\}$, for $j = 1, \ldots, M$. From the proof of Theorem 3.2 we know that $h \rightsquigarrow (b^1, \ldots, b^M, a^{-K}, \ldots, a^{-(M+1)}) =: \bar{h}$. Because of the special way we chose B (and because players sometimes mimic) we have $\bar{h} \rightsquigarrow \hat{h}$. □

It is possible to prove Theorem 3.3 with a smaller lower bound on the length of the memory by making full use of the presence of mimickers. We will not pursue that here. We just remark that for weakly acyclic games, the class of games considered in Young (1993), we could take $\underline{K} = 1$.

The learning process we considered implies that Bayesian players play best responses against past play. If a player knew that other players are following this process, he could do better by playing a strategy that is a best reply against a strategy profile, consisting of best responses for the other players against past play. Of course, we may have players who foresee that others are going to play best responses to best replies to past play. We could have even more sophisticated players. When we assume that in a class many different levels of sophistication are represented, we have a learning process with sophisticated players. (See also Milgrom and Roberts (1991).)

Formally, let h be a particular history and let $T^0(h) = \text{span}(h)$. Define recursively $T^{j+1}(h) = \text{span}(T^j(h) \cup B(\mathcal{B}^{ind}(T^j(h))))$. Since $T^{j+1}(h) \supset T^j(h)$ and A is finite, $T^\infty(h) = \text{span}(\cup_{j=0}^{\infty} T^j(h))$ is well-defined. Again, we define a whole set of transition matrices that correspond to learning processes with sophisticated players. We will denote this class by $\mathcal{P}^{soph}$, where $\mathcal{P}^{soph}$ is defined as follows.

Definition 3.4
Let $\mathcal{P}^{soph}$ denote the set of transition matrices P, that satisfy for all histories $h, \hat{h} \in H$,

$$P(h,\hat{h}) > 0 \quad \Leftrightarrow \quad \begin{cases} \hat{h} \textit{ is a successor of } h, \textit{ and} \\ r(\hat{h}) \in B(\mathcal{B}^{ind}(T^{\infty}(h))) \end{cases}$$

It is obvious that $\mathcal{P}^{soph} \subset \tilde{\mathcal{P}}$ and hence Theorem 3.2 is valid, also for this class. We can prove a stronger result: In the presence of sophisticated players we only need a memory of length one. The intuition for this result is that sophisticated players can do all the learning in their heads. They might foresee all the steps that needed to be executed in the case of no sophisticated players.

Theorem 3.4 *For all $K \geq 1$ and all Markov chains with a transition matrix $P \in \mathcal{P}^{soph}$ we have $Z \subset H$ is an ergodic set if and only if $Z = C^K$ for some minimal curb set C.*

Proof. For notational convenience we just give the proof for $K = 1$. Now $H = A$ and we can define $T^{\infty}(a)$ for all $a \in A$. Note that $T^{\infty}(a)$ is a curb set and hence there exists a minimal curb set $\bar{C} \subset T^{\infty}(a)$. If $\bar{a} \in \bar{C}$ then $P(a, \bar{a}) > 0$.

Note that if $a \in C$ for some minimal curb set C then $T^{\infty}(a) = C$. Hence, if $a, \bar{a} \in C$, then $P(a, \bar{a}) > 0$. □

The reader may have noticed that this sophisticated learning process has some similarities with the notion of rationalizability (Bernheim (1984) and Pearce (1984)). The difference is that rationalizability corresponds with a process of iterative elimination of strategies that are never best replies (starting with the whole space of strategy profiles) whereas our learning process implies the addition of best replies (starting from a history). The bounded memory of the players causes play to settle down in a *minimal* curb set.

The similarity of rationalizable and curb strategies has already been pointed out by Basu and Weibull (1991) and Balkenborg (1992): Call a set $C = \prod_{i=1}^{n} C_i$ *tight* if $B(\prod_{i=1}^{n} \Delta(C_i)) = C$. The maximal tight set is the set of rationalizable strategies, the minimal tight sets are just the minimal curb sets. In particular, every curb strategy is rationalizable.

3.5.2 Uncertain players

Consider the game from Figure 3.4.

	L	R
T	1,1	1,1
B	1,1	0,0

Figure 3.4.

This game has a unique curb set: it consists of all pure strategy profiles. When players behave as described by any of the learning processes they will regularly be playing (B, R)! This might seem a bit strange. It could not happen if the players were careful and only played undominated best replies. Then they would finally be playing only (T, L).

This example shows a drawback of the notion of minimal curb sets: They can contain strategies that are weakly dominated. Therefore let us recall from Basu and Weibull (1991) the notion of sets that are closed under undominated best replies. Let UB(s) denote the set of pure best replies against s that are not weakly dominated.

Definition 3.5 *A non-empty cartesian set $C = \prod_{i=1}^{n} C_i$ is closed under undominated best replies (or C is a curb* set) if $UB(\prod_{i=1}^{n} \Delta(C_i)) \subset C$. Such a set is called a minimal curb* set if it does not properly contain a set that is closed under undominated best replies. Strategies contained in minimal curb* sets are called curb* strategies.*

It is easy to adjust the learning process so that players will end up playing curb* strategies. Just replace 'best replies' by 'undominated best replies' and analogs of Theorems 3.2, 3.3 and 3.4 can be proved easily. On the level of Bayesian players this means that, although they have certain beliefs, they are not completely sure that these beliefs are "correct".[3] Therefore they should be careful and only play undominated best replies.

The approach taken above is a bit unsatisfactory since the uncertainty is not modeled. We will do that now. Remember the sampling procedure described in Section 3.3. Every time an individual is drawn from class V_i, he hears about L precedents concerning the way player j played this game before. This sample is transformed (by the fictitious play rule) into a belief s_{-i} from the L-grid distribution space $Gr^i(h, L)$, where h denotes the recent history of plays.

Now suppose that the final belief of this player is not necessarily s_{-i}, but some $\hat{s}_{-i}$ "close" to s_{-i}, reflecting the uncertainty of this player. This uncertainty may stem from the fact that the player realizes that he only draws a sample, and that s_{-i} is only a point

[3]The uncertainty of the players could stem from the fact that players may realize that other players have different samples. Anyway, sometimes players "are right" to be uncertain since it is possible that a history h is followed by the play of a, where $a \notin \text{span}(h)$.

estimate of the distribution of strategies. The final belief $\hat{s}_{-i}$ could be a draw from some "confidence interval" around s_{-i}. This draw might depend on personal characteristics, as well as on other external factors. We will just assume that $\hat{s}_{-i}$ is drawn from the uniform distribution over $B_\varepsilon(s_{-i}) = \{s'_{-i} \in S_{-i} | d^i_{max}(s_{-i}, s'_{-i}) \leq \varepsilon\}$, where $\varepsilon > 0$ is fixed[4] and where $d^i_{max}(s_{-i}, s'_{-i}) = \max_{a_{-i} \in A_{-i}} |s_{-i}(a_{-i}) - s'_{-i}(a_{-i})|$. Note that, for large L, the union of these intervals over all L-grid distributions induced by h, consists of all probability distributions close to $\prod_{j \neq i} \Delta(\pi_j(h))$.

What consequences does this have for our learning process? Or, in other words, what strategies will be played with positive probability after each possible history? Well, let $h \in H$ and let $a_i \in A_i$. Before we had that a_i was played with positive probability, whenever there was some $s_{-i} \in \prod_{j \neq i} \Delta(\pi_j(h))$ such that $a_i \in B_i(s_{-i})$. Now we have that a_i is played with positive probability, only if the stability region of a_i,

$$\mathrm{St}_i(a_i) = \{s_{-i} \in S_{-i} | a_i \in B_i(s_{-i})\},$$

has positive probability under the uniform distribution over $B_\varepsilon(\hat{s}_{-i})$, for some L-grid distribution $\hat{s}_{-i}$ induced by h. For sufficiently large L, this is equivalent to

$$s_{-i} \in \mathrm{cl}(\mathrm{int}(\mathrm{St}_i(a_i))), \tag{3.5.3}$$

for some $s_{-i} \in \prod_{j \neq i} \Delta(\pi_j(h))$, where $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ stand for closure and interior (in the topological space S_{-i}), respectively.

Note that if $s_{-i} \in \mathrm{int}(\mathrm{St}_i(a_i))$, then a_i is a best reply against each strategy in an open neighborhood of s_{-i}. Up to equivalence, a_i is then also the unique (and undominated) best reply against this neighborhood, and a_i is called a robust best reply against s_{-i}. If only (3.5.3) is satisfied, there is some non-empty open set close to s_{-i} against which a_i is the unique best reply, and we call a_i a semi-robust best reply against s_{-i}, which is denoted by $a_i \in \mathrm{SRB}_i(s_{-i})$. As opposed to robust best replies, semi-robust best replies always exist, and there may exist several semi-robust best replies against some s_{-i}, even if player i has no equivalent strategies. It is easy to see that semi-robust best replies are not weakly dominated. Similar to the case with the (undominated) best reply correspondence we define

Definition 3.6 (Balkenborg (1992))
A non-empty cartesian set $C = \prod_{i=1}^n C_i$ is closed under semi-robust best replies (or C

[4]We could take $\varepsilon = 1/L$ to reflect the intuition that bigger samples should result in smaller confidence intervals.

is a robust set) if $SRB(\prod_{i=1}^{n} \Delta(C_i)) \subset C$. Such a set is called a minimal robust set if it does not properly contain a set that is closed under semi-robust best replies.

The learning process where players are uncertain can be described by a Markov chain that is very similar to the ones we had before. Just replace 'best replies' by 'semi-robust best replies' and analogs of Theorems 3.2, 3.3 and 3.4 can be proved easily. Play will settle down in a minimal robust set.

For 'generic' normal form games the minimal curb, curb* and robust sets coincide with the persistent sets. Persistent sets consist of the extreme points of persistent retracts (Kalai and Samet (1984)). As a matter of fact, for games in which no player has equivalent strategies, the minimal robust sets coincide with the persistent sets (see Balkenborg (1992)). However, many normal form games are interesting because they are the normal form representation of an extensive form game, and these are not 'generic' in the class of normal form games. This is due to the fact that there may be strategies in the extensive form game that preclude some information sets (or subgames) from being reached. This implies that curb sets may differ from robust sets. To illustrate this difference consider the following example.

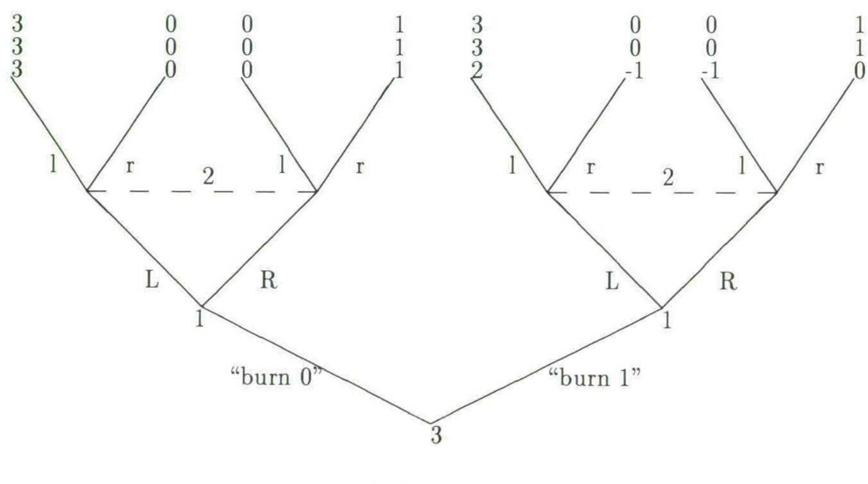

Figure 3.5.

Consider the game in Figure 3.5. Player 3 can decide to burn one unit before players 1 and 2 play a simultaneous move coordination game. Consider the strategy profile $a^{\text{ineff}} = (RR, rr, \text{"burn 0"})$. The singleton set containing this profile is persistent and robust: Player 3 has a unique best reply against a^{ineff}, namely "burn 0"; players 1 and 2 have a lot of (undominated) best replies against a^{ineff}, but they have a unique semi-robust best reply. In a small neighborhood outside $\{a^{\text{ineff}}\}$ players 1 and 2 have a unique best reply, since they have an interest in choosing the same action: in a small neighborhood player 1 plays 'R' with a very high probability, whether or not player 3 burnt something, and hence player 2 has to choose 'r', whether or not player 3 burnt something.[5] Since players 1 and 2 have a lot of (undominated) best replies against a^{ineff}, it is easy to see that $\{a^{\text{ineff}}\}$ is not curb or curb*. In fact, the only minimal curb (or curb*) set of this game consists of all strategy profiles yielding the payoff vector $(3,3,3)$. The latter set is also persistent and robust.

It seems that the learning processes introduced in this chapter are a bit peculiar in the case of extensive form games. In the process that leads to minimal curb sets, sometimes players are absolutely sure that a particular information set will not be reached. Therefore they are free to choose any action in this information set. On the other hand, if we add a little bit of uncertainty, players are still quite certain about the strategies that will be used, but they are also certain that all information sets will be reached with positive probability. Therefore they have to play a best reply against the strategy profile that they believe to be played almost certainly, in all information sets, although many of these information sets will not be reached if this strategy profile is indeed played.

These peculiarities are due to our assumption about the information that players have. In our learning process we assumed that players know the *strategies* played in the past. For extensive form games it makes more sense to assume that players only observe the *outcomes* of actual play, and that they may hold any beliefs about strategies in unreached information sets. We deal with this issue in Section 3.6.

3.5.3 Dependent beliefs

Throughout this chapter we assumed that a player's belief about the strategies of the other players is independent, i.e. is an element of S_{-i}. This was a consequence of the sampling procedure we described in Section 3.3. Players receive information about the strategies of the players individually. Moreover, if players realize that the players are deciding simultaneously and independently, then it is natural to have independent beliefs. There are however two problems concerning the independency of beliefs.

First of all, do players indeed decide independently? After all, the choices of all

[5]This result depends on the assumption of independent beliefs. See also Section 3.5.3.

players depend (via the samples) indirectly on the same recent history. History might act as a correlation mechanism. Secondly, our other interpretation of the learning process was that personal characteristics are important to form beliefs. All players expect that strategies that have not been played recently, will not be played, but different players may have different assessments of the probabilities with which the remaining strategies are played. In view of this interpretation, an individual from class V_i might have a dependent belief, i.e. an element of $\Delta(A_{-i})$. For instance, he might believe that the other players can correlate their strategies. It does not really matter whether or not the other players do correlate, what matters is that some individuals may believe that they do.

In this section we will examine the consequences of allowing players to have dependent beliefs. We will assume that the classes are very diverse: If h denotes the recent history and $a_i \in B_i(s^c_{-i})$ for some $s^c \in \Delta(\text{span}(h))$, then a_i will be played with positive probability. Again, we will define a whole set of transition matrices describing such learning processes. Let $\mathcal{B}^{dep}(h) = \{s^c \in \Delta(A) | \text{supp}(s^c) \subset \text{span}(h)\}$ denote the set of all dependent beliefs a player may have.

Definition 3.7
Let $\mathcal{P}^{dep}$ denote the set of transition matrices P, that satisfy for all histories $h, \hat{h} \in H$,

$$P(h,\hat{h}) > 0 \quad \Leftrightarrow \quad \begin{cases} \hat{h} \text{ is a successor of } h, \text{ and} \\ r(\hat{h}) \in B(\mathcal{B}^{dep}(h)) \end{cases}$$

Remark. Note that our definition of the transition matrices does not correspond to what one may call "correlated learning". Suppose that in a three player game player 3 observes that the other players played TL and BR in the last two periods. Then, under our assumption of dependent beliefs, it is possible that player 3 believes that TR and BL will be played, both with probability 1/2. One may feel that only beliefs of the form $\alpha TL + (1-\alpha)BR$ should be allowed. We do not know whether such "correlated learning" processes converge to some static set-valued solution concept.

We can prove a theorem similar to Theorem 3.2. Of course, the process will in general not converge to a minimal curb set, but to a cartesian set $F = \prod_{i=1}^{n} F_i$ that is minimal with respect to the following property: If $s^c \in \Delta(F)$ and $a_i \in B_i(s^c_{-i})$, then $a_i \in F_i$. Following Harsanyi and Selten (1988) we call such a set a primitive formation.[6]

[6]Harsanyi and Selten (1988) consider this concept in the agent normal form.

Theorem 3.5 *There exists $\underline{K} \in \mathbb{N}$ such that for all $K \geq \underline{K}$ and for every Markov chain with a transition matrix $P \in \mathcal{P}^{dep}$*
(i) If $Z \subset H$ is an ergodic set then $Z \subset F^K$ for some primitive formation F.
(ii) For every primitive formation F there exists exactly one ergodic subset $Z \subset F^K$.

We omit the proof because it is essentially the same as the proof of Theorem 3.2. We just have to observe that if F is a primitive formation and $a \in F$, then a_j is a best reply against some (dependent) belief concentrated on F.

Obviously, analogs of Theorems 3.3 and 3.4 to the case of dependent beliefs also exist. The same is true for the results of Section 3.5.2 on undominated best replies and semi-robust best replies. Analogous to curb* and robust sets we could define primitive* and robust formations. The reader should be aware, though, that the definition of semi-robustness needs to be adapted. For details, the reader is referred to Chapter 2.

3.6 Learning from outcomes

Throughout this chapter we assumed that players know the strategies that were used in the past. This assumption is reasonable when the players in the underlying game choose their actions simultaneously. But if the underlying game is in fact an extensive form game, it makes more sense to assume that players only observe the outcomes, i.e. the paths in the tree generated by the strategies. Consider for example the "burning money" game in Figure 3.5. Suppose player 3 chose to "burn 0" in the last period. How could he know how players 1 and 2 would have reacted to "burn 1"? In fact, he can't, although he may have some beliefs.

In this section we will consider the case where players only observe the outcomes in the recent past. We assume that all agents form expectations on the basis of observed outcomes, and that different agents within a pool may form different beliefs. We pose only one restriction on the beliefs: When a player is able to conclude from the observed outcomes that a particular strategy has not been played during the last K periods, then he expects it will not be played next period. As before, we assume that the classes are very diverse: As soon as strategy a_i is a best reply against some independent belief, satisfying this restriction, then a_i will be played with positive probability.

We will define a class of transition matrices that correspond to such a "learning from outcomes" process, and we denote this class by $\mathcal{P}^{out}$. Before we can do so, we need some notation.

Let g be an extensive form game. Let $\mathcal{O}$ denote the set of outcomes (i.e. paths in the tree from the root to an endpoint) and let $o : A \to \mathcal{O}$ be the mapping that assigns to a pure strategy combination the outcome it generates. We will assume that there are no moves of Nature in g, since this mapping is not well-defined if there are. For a history $h = (a^{-K}, \ldots, a^{-1})$, let $\text{outc}(h) = \{o(a^{-K}), \ldots, o(a^{-1})\}$. Note that $\text{outc}(h)$ summarizes the information a player has. Let $\text{cons}_i(h) = \{a_i \in A_i | \exists a_{-i} \in A_{-i} \text{ s.t. } o((a_i, a_{-i})) \in \text{outc}(h)\}$ denote the set of strategies of player i that are consistent with the observed outcomes. Let $\text{cons}(h) = \prod_{i=1}^{n} \text{cons}_i(h)$.

Definition 3.8

Let $\mathcal{P}^{out}$ denote the set of transition matrices P, that satisfy for all histories $h, \hat{h} \in H$,

$$P(h, \hat{h}) > 0 \quad \Leftrightarrow \quad \begin{cases} \hat{h} \text{ is a successor of } h, \text{ and} \\ r(\hat{h}) \in B(\mathcal{B}^{ind}(cons(h))) \end{cases}$$

In general, it is not true that play will settle down in minimal curb sets. Note that $\text{cons}(h) \supset \text{span}(h)$. This implies that if $P \in \mathcal{P}$ and $P(h, \hat{h}) > 0$, then $P^{out}(h, \hat{h}) > 0$ for all $P^{out} \in \mathcal{P}^{out}$. Using part of the proof of Theorem 3.2, it follows that, if K is large enough, for every history h and every $P^{out} \in \mathcal{P}^{out}$, there exists a curb history $\tilde{h}$ such that $h \rightsquigarrow \tilde{h}$. The problem is that there might exist a history $\hat{h}$, which is not a curb history, such that $\tilde{h} \rightsquigarrow \hat{h}$. This might even happen in 'generic' extensive form games, as the game from Figure 3.6 shows.

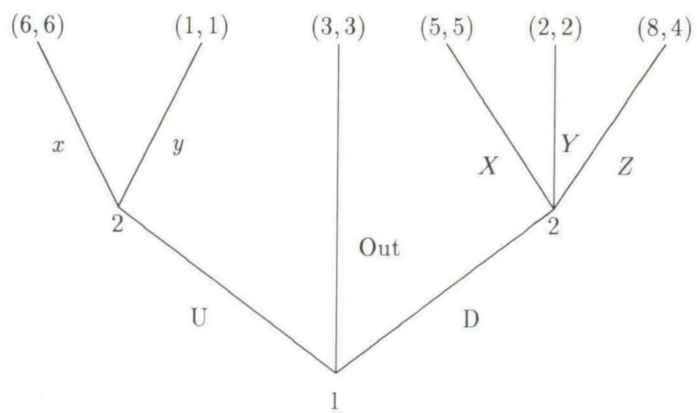

Figure 3.6.

This game has a unique minimal curb set, namely $\{U, D\} \times \{xX, xY, xZ, yX\}$. However, suppose that in the recent (curb) history the strategy combinations (D, xY) and

(U, yX) were played. Hence, player 1 observes (amongst other things) the outcomes DY and Uy. He might believe that the strategy yY was played, and will be played again next period. If he does so, he will choose 'Out', which is not a curb strategy.

The above example seems to suggest that there is no hope to obtain a result like Theorem 3.2 in the case of learning from outcomes. There are however two classes of games for which such an analog does exist. The first class consists of the extensive form games without moves of nature, where each player has only one information set at which he has to make a choice. For obvious reasons we call such a game an agent normal form game without moves of nature, and we denote the class by ANF. The second class of games consists of those games g that have the property that any minimal curb set C of g corresponds to a single outcome, i.e. the set $\{o(c)|c \in C\}$ is a singleton. We denote this class by SCO (single curb outcome). Examples of these games are shown in Figures 3.1b, 3.1c and 3.5.

To prove the above claims we just need to show that $\mathcal{P}^{out} \subset \tilde{\mathcal{P}}$, where $\tilde{\mathcal{P}}$ is as defined at the beginning of Section 3.5. Part (3.5.1) follows from $\text{span}(h) \subset \text{cons}(h)$, part (3.5.2) follows from the next lemma.

Lemma 3.2 *Let $g \in ANF$ or $g \in SCO$ and let C be a minimal curb set of g. Then*

$$h \in C^K \quad \Rightarrow \quad cons(h) \subset C$$

Proof. First consider the case $g \in$ ANF. Let j be a player. If there is an outcome $o(a^{-m}) \in \text{outc}(h)$ that does not intersect j's information set, then it follows that $B_j(a^{-m}) = A_j$. This implies that $C_j = A_j \supset \text{cons}_j(h)$. If there is no such outcome, all outcomes intersect j's information set and $\text{cons}_j(h) = \pi_j(h) \subset C_j$. Hence, $\text{cons}(h) \subset C$.

Now consider the case $g \in SCO$. Let $\bar{a} = r(h)$. Now we have $\text{outc}(h) = \{o(\bar{a})\}$. Let j be a player and suppose $a_j \in \text{cons}_j(h)$. In any information set of j that intersects $o(\bar{a})$, a_j picks the same action as $\bar{a}_j$, since a_j is consistent with h. Since $g \in SCO$, we have that $\bar{a}_j$ is a best reply against $\bar{a}_{-j}$. But this implies that a_j is a best reply against $\bar{a}_{-j}$ as well, and hence $a_j \in C_j$. □

The reader can check that there are also analogs of Theorems 3.3 and 3.4 to the case where players learn from outcomes. The definition of a mimicker needs to be adapted, since players don't observe strategies. We may assume that mimickers choose at random a strategy from the set of strategies that are consistent with (some of) the observed outcomes. There is also an analog of Theorem 3.5, where players' beliefs are not independent. There are however no analogies for the results of Section 3.5.2 on the refined notions of undominated best replies or of semi-robust best replies. This is due to the fact that strategies that are consistent with a curb* history, may be weakly dominated. The game of Figure 3.6 shows an example of such a case: The only curb* strategy is (U, xX), but xY and xZ are consistent with the curb* outcome.

3.7 Learning and experimentation

In many papers on learning, experimentation plays a prominent role. (See e.g. Kandori, Mailath and Rob (1993), Samuelson (1994), Young (1993) and Fudenberg and Kreps (1988)).

In Young (1993), Samuelson (1994) and Kandori et al. (1993) the possibility of experimentation (or mistakes, or mutations) implies that the Markov chain describing the learning process becomes irreducible, and hence has a unique stationary distribution. By taking the limit as the experimentation rate tends to zero, one stationary distribution of the unperturbed process is selected. In Young (1993) and Kandori et al. (1993) this yields typically a unique so called stochastically stable state because they consider a special class of games. Samuelson (1994) considers games with alternative best replies and then the support of the limit distribution consists usually of one or more line segments.

It turns out that the introduction of experimentation does not change the results of this chapter, at least not for two person games. If a two person game has multiple minimal curb sets, experimentation will not yield the selection of a particular one: the limiting distribution puts positive weight on all states that are ergodic under the unperturbed process. The intuition behind this result is that only one mistake by one player is necessary in order to move the system from one ergodic set to another. When the game has more than two players, it might happen that a particular minimal curb set is selected. One can characterize the selected minimal curb set graph-theoretically.

In order to prove these results formally, we would have to recall the essential definitions and theorems from Young (1993). We refer the reader to the original paper for a formal treatment. We will just illustrate the result by means of an example.

Consider again the coordination game from Figure 3.1a. As we have seen before this game has two minimal curb sets, $\{(T, L)\}$ and $\{(B, R)\}$. Suppose the system is in state $h^{TL} = (TL, \ldots, TL)$ and player 1 makes a mistake and plays B. Since sampling occurs with replacement, player 2 may receive a draw containing many B's, in which case he will play R. It may happen that from then on player 1 receives draws with many R's while player 2 keeps drawing many B's. It follows that, after the initial mistake, the system can move to $h^{BR} = (BR, \ldots, BR)$, without making any further mistakes. Hence, only one mistake is needed to move the system from h^{TL} to h^{BR}. Similarly, only one mistake is needed to move the system from h^{BR} to h^{TL}. Since the mistake probabilities are of the same order, the limiting distribution puts positive weight on both ergodic states.

This result is in contrast with Young (1993). In Young (1993) the players also have information about play in the recent history: Every player draws a sample of m plays out of the plays of the most recent K periods, but without replacement. Then players play a best reply in a fictitious play fashion. Consider again the coordination game from Figure 3.1a. Suppose that the system is in state h^{TL} and that player 1 makes a mistake and plays B. If no further mistakes occur the system will move back to h^{TL}, if the sample size is at least 2: Since sampling occurs without replacement, every sample contains at least as many T's as B's. and player 2 will always play L (unless he makes a mistake). It is easy to see that in this example at least $3m/4$ mistakes are needed to move the system from h^{TL} to h^{BR}, while only $m/4$ mistakes are needed to move the system in the other direction. It follows that h^{TL} is the unique stochastically stable state.

3.8 Concluding remarks

We have considered learning processes where the players have a bounded memory and play best replies against past play. The importance of the bounded memory can be elucidated by comparing our learning process with Milgrom and Roberts (1991). In general they consider games with compact strategy sets that are played continuously. Translated to the context of a two player finite normal form game which is played repeatedly at discrete points in time, they define a sequence of plays $\{a(t)\}_{t=0}^{\infty}$ to be *consistent with adaptive learning* if for all $\hat{t}$, there exists a $\bar{t}$ such that for all $t \geq \bar{t}$, $a(t+1) \in B(\mathcal{B}^{ind}(\{a(\hat{t}), a(\hat{t}+1), \ldots, a(t)\}))$. We could similarly define this sequence to be *consistent with learning with bounded memory*, if there exists $K \in \mathbb{N}$ such that for all t, $a(t+K) \in B(\mathcal{B}^{ind}(\{a(t), a(t+1), \ldots, a(t+K-1)\}))$. This definition illustrates the similarity between this chapter and Milgrom and Roberts (1991).

Consider for example the pure coordination game of Figure 3.1a. The sequence $TR, BL, TR, BL, TR, \ldots$ satisfies both definitions of consistency. However, the finiteness of the memory and of the strategy space allows us to obtain a finite Markov chain, from which we can compute that the probability of obtaining the above sequence is zero: Only sequences with tails $TL, TL, TL, \ldots$ or $BR, BR, BR, \ldots$ are obtained with positive probability.

Milgrom and Roberts (1991) show that sequences that are consistent with adaptive learning will eventually lie within the set of serially undominated strategies, which is a superset of the set of rationalizable strategies. They give some examples of games with strategic complementarities where this set is a singleton, which implies that these sequences must converge to the unique equilibrium. We get the same results in these games because the set of curb strategies is a subset of the set of rationalizable strategies. But we get similar results in some games where the set of rationalizable strategies is big. In every game that has a unique and strict equilibrium $\bar{a}$, $\{\bar{a}\}$ is the unique minimal curb set. Hence, in such games our learning process leads the players (with probability 1) to the unique equilibrium (Corollary 3.2). An example of such a game is given in Figure 3.3, where all strategies are rationalizable.

Another example is the discretized version of the following three player Cournot oligopoly game. Player i chooses to produce q_i at zero costs to maximize $q_i(D - q_1 - q_2 - q_3)$. The unique (and strict) equilibrium is $(D/4, D/4, D/4)$. The set of rationalizable strategies is $[0, D/2] \times [0, D/2] \times [0, D/2]$.

Chapter 4

Multi-Sided Pre-Play Communication by Burning Money

4.1 Introduction

Given a game in normal form, Ben-Porath and Dekel (1992) investigate the consequences of allowing some players to signal future actions by incurring costs before the game is played. They consider equilibria that survive repeated elimination of weakly dominated strategies. Their main result is that, in a certain class of two person games, if only one player can signal, then repeated elimination of weakly dominated strategies selects her most preferred outcome. Moreover, the player does not have to incur any cost to achieve this.

While this result is nice, it has some important limitations. First of all, Ben-Porath and Dekel consider only two player games. As we will show later by example, repeated elimination of weakly dominated strategies need not work in games with more than two players. Furthermore, they show that if both players have the opportunity to signal (simultaneously), then signaling future actions is not possible, not even if the game has common interests. Finally, they need that a player can burn a considerable amount of money.

We consider a more general case. We extend n-person games by allowing k of the players to signal future actions by incurring costs. In order to obtain results similar to those of Ben-Porath and Dekel we work with a stronger solution concept. Van Damme (1989) showed that stable equilibria do not necessarily lead to efficient outcomes. We show that, if the notion of curb or curb* retracts (Basu and Weibull (1991)) is used, then signaling future actions is possible. Roughly speaking, if the players that can burn money have common interests in the underlying game, then curb and curb* retracts select their preferred outcome. Furthermore, in all models of pre-play communication, no costs are actually incurred. Moreover, if there are two players then the notion of persistence (Kalai and Samet (1984)) gives the same results.

In the next section we will consider a simple example of a two person game in which only one player can signal. From this example the reader can develop some feeling for the solution concept we employ. Moreover, this example shows a difference between our approach and that of Ben-Porath and Dekel's. At the same time it shows that it is important that messages are costly.

In Section 4.3 the formal model is introduced. In Section 4.4 we prove the theorem for the concepts of curb and curb* retracts for the general case. In Section 4.5 we consider the persistent retracts in the special case of two person games. Moreover, it is shown that in games with more than two players persistence need not work. Section 4.6 examines the consequences of analyzing the game in the agent normal form instead of the normal form. We close this chapter with some concluding remarks.

4.2 An example: the Battle of the Sexes

Consider the Battle of the Sexes game represented by the normal form in Figure 4.1. The woman (the row player) prefers to go to a soccer match ('S'), the man (the column player) prefers to go to the theater ('t').

	s	t
S	9,5	0,4
T	4,4	6,7

Figure 4.1: the Battle of the Sexes.

Suppose the woman can send one of two messages, m^0 or m^1. Message m^0 is costless and message m^1 costs c. Later we will consider the cases $c = 0$, $c = 1$ and $c = 2$. Let m^iE denote the woman's strategy when she sends m^i and visits event E. Let e_0e_1 denote the man's strategy when he goes to event e_i if he receives message m^i. Then the game with one-sided pre-play communication can be represented by the normal form in Figure 4.2.

	ss	st	ts	tt
m^0S	9,5	9,5	0,4	0,4
m^0T	4,4	4,4	6,7	6,7
m^1S	$9-c,5$	$-c,4$	$9-c,5$	$-c,4$
m^1T	$4-c,4$	$6-c,7$	$4-c,4$	$6-c,7$

Figure 4.2.

First we will consider Ben-Porath and Dekel's approach. Consider the case $c = 2$. Action m^1T is (weakly) dominated by m^0T. So the man should go to the soccer match if he receives message m^1. This means that st and tt are dominated. If the woman knows that the man will play ss or ts, then the action m^0T is dominated by m^1S. So the man knows that the woman will go to the soccer match, hence, he should go to the soccer match also (st is dominated by ss). If the woman knows that, she will send m^0 and go to the soccer match (m^1S is dominated by m^0S). Hence, repeated elimination of weakly dominated strategies selects the outcome where both go to the soccer match and no money is actually being burnt. Notice that this reasoning does not go through if $c = 0$ or if $c = 1$. In these cases no player has a dominated strategy. For example, if $c = 1$, m^1T can only be dominated by a strategy that puts at least weight $\frac{5}{6}$ on m^0T, and at least weight $\frac{1}{5}$ on m^0S.

In this chapter we will use the concepts of curb, curb* and persistent retracts. These were introduced in Chapter 2. Now look at our example. Suppose $R_1 \times R_2$ is a retract that is closed under best replies. It is easy to check that we have the following implications (for all values of c under consideration).

$$\begin{array}{lllllllll} m^1T \in R_1 & \Rightarrow & tt \in R_2 & \Rightarrow & m^0T \in R_1 & \Rightarrow & ts \in R_2 & \Rightarrow \\ m^1S \in R_1 & \Rightarrow & ss \in R_2 & \Rightarrow & m^0S \in R_1 & \Rightarrow & ss, st \in R_2 \end{array}$$

In the cases with $c > 0$ the set of strategy profiles where the woman plays m^0S and the man mixes between ss and st is closed under best responses. The series of implications shows that this set is the unique minimal one.

If $c = 0$ the series of implications continues:

$$ss, st \in R_2 \quad \Rightarrow \quad m^1S \in R_1 \quad \Rightarrow \quad ts \in R_2.$$

Since m^0T and m^1T are best replies against $\frac{1}{2}st + \frac{1}{2}ts$, we conclude that the only set that is closed under best responses is the set of all strategy profiles.

Hence, if talk is costless, then on the basis of curb no sharp predictions can be made in this game.[1]

4.3 The model

In this section we will formally set up the model of multi-sided pre-play communication in n-person games. We will use notation as introduced in Section 1.3. For the definitions of curb, curb* and persistent retracts the reader is referred to Chapter 2.

Let $g = (A_1, \ldots, A_n, u_1, \ldots, u_n)$ be an n-person game with player set N. We split up the set of players into C and D. C is the set of players that can send a message in the pre-play communication stage (communicating players); D is the set of players that cannot (dumb players). Note that the players in D are dumb but not deaf. It is important that they can hear. We assume that C is not empty. D may be empty.

We will assume

Assumption 4.1 *There exists $a^* \in A$ such that $u_i(a^*) > u_i(a)$, for all $i \in C$ and all $a \neq a^*$ and such that $u_j(a^*) > u_j(a^*_{-j}, a_j)$ for all $j \in D$ and all $a_j \neq a^*_j$.*

Notice that if $C = N$ then g is a common interest game (Aumann and Sorin (1989)). Assumption 4.1 says that the game has a strict Nash equilibrium that gives all communicating players their highest payoff.

We assume that all communicating players have the same set of messages $M = \{m^0, m^1, \ldots, m^L\}$ at their disposal and that the cost of sending message m^p is $c(m^p)$ for all of them. These assumptions are made for notational convenience only. They do not play a role in the results we will derive. We assume that $c(m^0) = 0$, $c(m^1) > 0$ is 'small' and $c(m^p) > c(m^1)$ for all $p > 1$. In fact, $c(m^1)$ is so small that

$$c(m^1) < u_i(a^*) - u_i(a) \qquad \text{for all } i \in C \text{ and all } a \neq a^*.$$

This implies that any communicating player that can induce the play of a^* by sending message m^1 will do so, unless sending m^0 induces the play of a^* also.

[1] Blume (1993b) shows that the equilibria in a curb retract of the one-sided cheap talk game select the equilibrium preferred by the sender, provided that the risk associated with this equilibrium in the underlying game is sufficiently low. Indeed, if the payoff to (S, t) is changed from '0, 4' into '2, 4' (which reduces the risk of (S, s)), then the unique curb retract does not contain m^0T, m^1T or tt, and all equilibria in the curb retract yield $9, 5$.

We denote the game with pre-play communication by $P_C(g) = (T_1, \ldots, T_n, v_1, \ldots, v_n)$, where

$$\begin{aligned} T_i &= \{m_i f_i | m_i \in M \text{ and } f_i : M^{C\setminus\{i\}} \to A_i\} \quad (i \in C) \\ T_j &= \{f_j | f_j : M^C \to A_j\} \quad (j \in D) \end{aligned}$$

(We use the following convention if $C = \{i\}$: $f_i : M^{\emptyset} \to A_i$ corresponds to a single element of A_i.) With some abuse of notation we have for $mf \in T = \prod_{l=1}^{n} T_l$,

$$\begin{aligned} v_i(mf) &= u_i(f(m)) - c(m_i) \quad (i \in C) \\ v_j(mf) &= u_j(f(m)) \quad (j \in D) \end{aligned}$$

Remark: The way we have defined the strategy space here means that we are looking at the reduced normal form (as did Ben-Porath and Dekel). In the normal form a communicating player's strategy also depends on his own message. It does not matter for our results whether we analyze the normal form or the reduced normal form. However, in the normal form the communicating players have a lot of equivalent strategies and this makes the proof for persistence quite tedious. We could also look at the agent normal form (Selten (1975)). We will do that in Section 4.6.

4.4 Results for the general case

In this section we will prove the theorem for the curb and curb* retracts. We will deal with the general case with n players among which the players in C can communicate.

Theorem 4.1 *$P_C(g)$ has a unique curb (curb*) retract, and every strategy profile in this retract yields player l $u_l(a^*)$ ($l \in N$).*

Proof. We will denote a vector with all coordinates equal to m^0 by $\widehat{m^0}$. (Depending on whether this vector is in the domain of a dumb or of a communicating player, it has $\#C$ or $\#C - 1$ coordinates. No confusion will result.) Define

$$F = \{mf \in T | m = \widehat{m^0}\ ,\ f_l(\widehat{m^0}) = a_l^* \text{ for all } l \in N\}$$

Notice that F consists of all pure strategy profiles that yield the payoff vector $u(a^*)$.

We will show that every curb* retract has a non-empty intersection with F. If this is the case then every curb* retract is contained in $\bar{R} = \prod_{l=1}^{n} \Delta(F_l)$, since $\bar{R}$ is closed under undominated best replies. Since every curb retract contains a curb* retract, we know then that every curb retract has a non-empty intersection with $\bar{R}$. Since $\bar{R}$ is closed under best replies itself, any curb retract is contained in $\bar{R}$. Since all strategy profiles in $\bar{R}$ yield the payoff $u(a^*)$, it follows that $\bar{R}$ is the unique curb retract, and that the unique curb* retract is obtained from $\bar{R}$ by deleting every strategy that has pure weakly dominated strategies in its support. Then the proof has been completed.

Let R be a curb* retract and let $\bar{m}\bar{f} \in R \setminus F$. Now player l has a lot of undominated best replies against $(\bar{m}\bar{f})_{-l}$. This is so because a player has a lot of freedom in how to react on tuples of messages that he does not receive. In particular, there exists $m'f' \in R$ such that (i) for all $i \in C$ and all $m_{-i} \neq \bar{m}_{-i}$, $f'_i(m_{-i}) = a^*_i$, and (ii) for all $j \in D$ and all $m \neq \bar{m}$, $f'_j(m) = a^*_j$.

If $m'f' \in F$ we are ready, because then $F \cap R \neq \emptyset$.

So suppose $m'f' \notin F$.

Case 1: $\bar{m} \neq \widehat{m^0}$. There exists $i \in C$ with $\bar{m}_i \neq m^0$. All best replies for i against $m'f'$ involve sending m^0. In particular, if $f''_i(m) = a^*_i$ for all m, then $m^0 f''_i$ is an undominated best reply to $m'f'$ and hence contained in R_i. For each $j \in C \setminus \{i\}$, all best replies against $z = (m^0 f''_i, m'f'_{-i})$ involve sending m^0. In particular, if $f''_j(m) = a^*_j$ for all m, then $m^0 f''_j$ is an undominated best reply to z and, hence, it is contained in R_j. Thus $F \cap R \neq \emptyset$.

Case 2: $\bar{m} = \widehat{m^0}$. Let $i \in C$. All best replies for i against $m'f'$ involve sending m^1. Hence, there exists $\bar{\bar{m}}\bar{\bar{f}} \in R \setminus F$ with $\bar{\bar{m}} \neq \widehat{m^0}$. This brings us back to case 1. □

4.5 Further results for two player games

Let us first state the following lemma which will prove to be useful in the two player case. The proof is relegated to the Appendix.

Lemma 4.1 *Let $g = (A_1, A_2, u_1, u_2)$ be a game. For $F = F_1 \times F_2 \subset A_1 \times A_2$, let $H_i = A_i \setminus F_i$. If (1) for all $f, f' \in F$, $u_i(f) = u_i(f')$, (2) for all $f \in F$ and $h_i \in H_i$, $u_i(h_i, f_j) < u_i(f)$, and (3) every persistent retract has a non-empty intersection with F, then every persistent retract is contained in $\bar{R} = \Delta(F_1) \times \Delta(F_2)$.*

From now on we will assume

Assumption 4.2 *No player has equivalent strategies in the game with pre-play communication.*

Then we have that the extreme points of every persistent retract of this game are pure strategies. For a proof see Chapter 2.

4.5.1 One-sided pre-play communication

First we will consider the case with one-sided pre-play communication, hence $C = \{1\}$. Assumption 1 is now equivalent to saying that the underlying game has a strict Nash equilibrium yielding player 1 a higher payoff than any other strategy profile.

Theorem 4.2 . *Every strategy profile in a persistent retract of $P_{\{1\}}(g)$ yields player l $u_l(a^*)$ $(l \in N)$.*

Proof. Let R be a persistent retract and let $ma_1 \in R_1$.
Case 1. $m \neq m^0$. Then there exists $f \in R_2$ with $f(m^0) = a_2^*$. This is so because all best replies against $(1-\varepsilon)ma_1 + \varepsilon m^0 a_1^*$ have this property. This implies that $m^0 a_1^* \in R_1$, since it is the unique best reply against f.
Case 2. $m = m^0$ and $a_1 \neq a_1^*$. Using the same trick as before we see that there exists $f \in R_2$ with $f(m^1) = a_2^*$. If $f(m^0) = a_2^*$, then $m^0 a_1^* \in R_1$. If $f(m^0) \neq a_2^*$, then $m^1 a_1^* \in R_1$. This brings us back to case 1.

Hence, $m^0 a_1^* \in R_1$. It is easy to check that the set of extreme points of

$$F = \{m^0 a_1^*\} \times \Delta(\{f : M \to A_2 | f(m^0) = a_2^*\})$$

satisfies assumptions (1), (2) and (3) of Lemma 4.1.

Then it follows from Lemma 4.1 that $R \subset F$. Notice that every strategy profile in F yields player l $u_l(a^*)$ and the theorem has been proved. □

4.5.2 Two-sided pre-play communication

Now we turn our attention to the two-sided pre-play communication game, i.e. $C = \{1, 2\}$. Recall that Assumption 4.1 now says that the underlying game has common interests.

Theorem 4.3 *Every strategy profile in a persistent retract of $P_C(g)$ yields player l a payoff of $u_l(a^*)$ $(l \in N)$.*

Proof. Let R be a persistent retract and let $mf_i \in R_i$. Let $j = 3 - i$.
Case 1. $m \neq m^0$. Then there exists $m' f_j \in R_j$ with $f_j(m^0) = a_j^*$. Hence, there exists $m^0 f_i' \in R_i$ with $f_i'(m^0) = a_i^*$.
Case 2. $m = m^0$ and $f_i(m^0) \neq a_i^*$. Then there exists $m' f_j \in R_j$ with $f_j(m^1) = a_j^*$. If $f_j(m^0) = a_j^*$, then there exists $m^0 f_i' \in R_i$ with $f_i'(m^0) = a_i^*$. If $f_j(m^0) \neq a_j^*$, then there exists $m^1 f_i'' \in R_i$. This brings us back to case 1.

Let $F_i = \{m^0 f | f(m^0) = a_i^*\}$. The two cases showed that $F_i \cap R_i \neq \emptyset$. It is easy to check that F_1 and F_2 satisfy assumptions (1), (2) and (3) of Lemma 4.1. Hence, any persistent retract is contained in $\Delta(F_1) \times \Delta(F_2)$. Notice that any strategy profile in this set yields player l $u_l(a^*)$ and the theorem has been proved. □

4.5.3 A three player counterexample

The following example shows that persistence need not work in games with more than two players.

Example. Let $n = 3$, $C = \{3\}$. Player 1 chooses between rows R_1 and R_2, player 2 chooses between columns C_1 and C_2 and player 3 chooses X. (Player 3 has only one action.) The payoffs are given in Figure 4.3.

	C_1	C_2
R_1	2,2,2	0,0,0
R_2	0,0,0	1,1,1

Figure 4.3.

Now consider the following strategy profile: players 1 and 2 play their second strategy after all messages and player 3 sends the costless message and then 'chooses' X. This yields the strictly dominated payoff vector $(1, 1, 1)$. Nevertheless, this strategy profile is a persistent retract as a singleton and, hence, a persistent equilibrium. In a small neighborhood of the retract player 3 has a unique best reply. Players 1 and 2 have a lot of best replies against the retract, but in a small neighborhood outside the retract they have a unique best reply. This is due to the fact that players 1 and 2 have an interest in choosing the same action: in a small neighborhood player 1 plays R_2 with a very high probability, after any message, and hence player 2 has to choose C_2, after any message.

Repeated elimination of weakly dominated strategies does not work either. In this example the messages are not important to signal the intended action of player 3, since he has only one action. Hence, our results are not related to forward induction. The reason that the dummy player has an effect on the outcome (when curb or curb* is applied), is that he is willing to try a different (and possibly more costly) message, whenever the first two players fail to coordinate on the efficient equilibrium.

4.6 Agent normal form

As remarked at the end of Section 4.3, we have analyzed the pre-play communication game in the reduced normal form. In this section we will look at the agent normal form (Selten (1975)). The agent normal form is obtained by splitting every player into several agents. Each agent corresponds to one information set of that player and each agent has the same utility function as that player. In the agent normal form different agents of the same player take their decisions independently.

Consider for example the one-sided pre-play communication game of the Battle of the Sexes of Section 4.2. The extensive form of that game (when $c = 2$) is given in Figure 4.4.

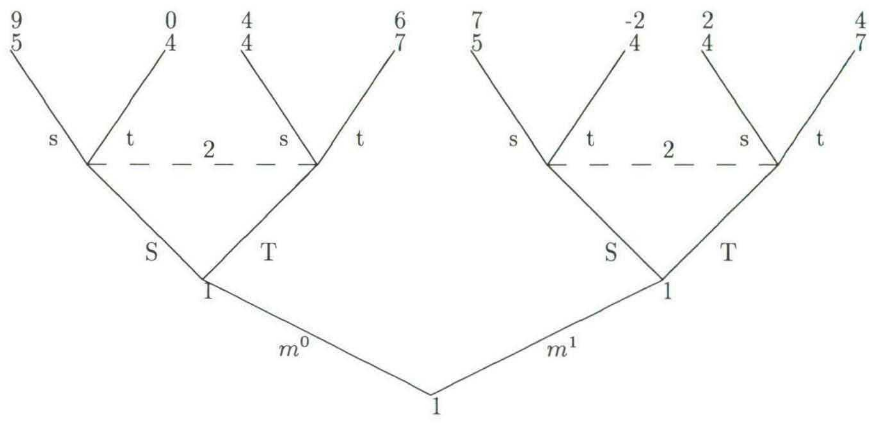

Figure 4.4.

Player 1 has three information sets, namely $\emptyset$, $\{m^0\}$ and $\{m^1\}$. So in the agent normal form there are three agents for player 1; call them 1_0, $1(m^0)$ and $1(m^1)$. Player 2 has two information sets, namely $\{m^0S, m^0T\}$ and $\{m^1S, m^1T\}$. Call the agents $2(m^0)$ and $2(m^1)$. So the agent normal form of this game has 5 agents (or players).

It is well known by now that a lot of refinements of the Nash equilibrium concept may give different results in the normal form and in the agent normal form.

In the agent normal form of the game in Figure 4.4 no agent has a weakly dominated strategy. Hence, repeated elimination of weakly dominated strategies does not work in the agent normal form.

For persistence we have similar problems. Consider the strategy profile where agent 1_0 sends m^0, agents $1(m^0)$ and $1(m^1)$ play T and agents $2(m^0)$ and $2(m^1)$ play t. One can check that this profile is a persistent retract as a singleton.

It is not surprising that persistence and repeated elimination of weakly dominated strategies give different results in the agent normal form and in the normal form. By splitting up the players into agents we have made a five player game out of a two player game. We have already seen that these solution concepts give different results in games with more than two players.

In principle we could have the same problems with curb and curb*. In general it is not true that the curb and curb* retracts in the normal form correspond to those in the agent normal form. However, the class of games considered in this chapter is special. We have

Theorem 4.4 *The agent normal form of $P_C(g)$ has a unique curb (curb*) retract. All strategy profiles in this retract yield every agent of player l, $u_l(a^*)$ $(l \in N)$.*

Proof. For $i \in N$ let $i(m)$ denote the agent of player i with information set $\{m\}$. For $i \in C$ let i_0 denote the agent of player i that sends a message. The number of agents is $\#C + (\#N) \times (\#M)^{\#C}$. Let $\mathcal{A}$ be the set of agents. A strategy profile is denoted by ma. Let $\widehat{m^0}$ be the vector with $\#C$ coordinates, all equal to m^0.

Define

$$F = \{ma | m = \widehat{m^0}\ ,\ a_{i(\widehat{m^0})} = a_i^* \text{ for all } i \in N\}$$

F consists of all pure strategy profiles that yield the payoff vector $u(a^*)$.

We will show that every curb* retract has a non-empty intersection with F. If this is the case then every curb* retract is contained in $\bar{R} = \prod_{j \in \mathcal{A}} \Delta(F_j)$, since $\bar{R}$ is closed under undominated best replies. Since every curb retract contains a curb* retract, we know then that every curb retract has a non-empty intersection with $\bar{R}$. Since $\bar{R}$ is closed under best replies itself, any curb retract is contained in $\bar{R}$. But then the theorem is proved since every strategy profile in $\bar{R}$ yields all agents of player l $u_l(a^*)$.

Let R be a curb* retract and let $\bar{m}\bar{a} \in R \setminus F$.

Case 1. $\bar{m} \neq \widehat{m^0}$.

For all agents $i(m)$ with $m \neq \bar{m}$ we have that a_i^* is an undominated best reply against $\bar{m}\bar{a}$. Hence, $\bar{m}\bar{a}_{i(\bar{m})}a^*_{-i(\bar{m})} \in R$. An agent i_0 with $\bar{m}_{i_0} \neq m^0$ has a unique best reply against this profile, namely to send m^0. Hence, $m^0\bar{m}_{-i_0}a_{i(\bar{m})}a^*_{-i(\bar{m})} \in R$. Against this strategy, each agent j_0 has a unique best reply, namely to send m^0. Now it is shown that $R \cap F \neq \emptyset$.

Case 2. $\bar{m} = \widehat{m^0}$.

For all agents $i(m)$ with $m \neq \bar{m}$ we have that a_i^* is an undominated best reply against $\bar{m}\bar{a}$. Hence, $\bar{m}\bar{a}_{i(\bar{m})}a^*_{-i(\bar{m})} \in R$. Every agent i_0 has a unique best reply against this strategy, namely to send m^1. Hence, there is a strategy profile $\bar{\bar{m}}\bar{\bar{a}} \in R$ with $\bar{\bar{m}} \neq \widehat{m^0}$. This brings us back to case 1. □

4.7 Conclusion and related literature

We have generalized the result of Ben-Porath and Dekel into two directions. We consider games with more than two players, and there may be more than one player that has the possibility to burn money. We have shown that curb and curb* retracts select the equilibrium that is preferred by the people that have the possibility to burn money.

A class of related results can be found in some of the literature on cheap talk. This literature usually restricts attention to two person games, but it also applies set-valued solution concepts. Blume (1993b) shows that all equilibria (but not all strategy profiles!) in a curb retract of the game with one-sided pre-play communication, select the outcome preferred by the sender, if the risk associated with this equilibrium is not too high. Matsui (1991) shows that cyclically stable sets (see also Chapter 2) select the optimal outcome in common interest games with two-sided pre-play communication. Kim and Sobel (1991) apply the concept of an equilibrium evolutionarily stable set (Swinkels (1992), see also Chapter 2). They obtain efficiency in common interest games when the players speak consecutively. However, they show that inefficiency may occur in common interest games (but not in equilibrium common interest games), when the players speak simultaneously.

Appendix

Proof of Lemma 4.1. Let R be a persistent retract, and let $V_1 \times V_2 \supset R$ be an open neighborhood that is absorbed by R. Let $x_i = u_i(f)$ for (all) $f \in F$ and define

$$\begin{aligned}
m_i^+ &= \max u_i(a) \\
m_i^- &= \min u_i(a) \\
\delta(h_1) &= x_1 - \max_{f_2 \in F_2} u_1(h_1, f_2) \quad (h_1 \in H_1) \\
\delta(h_2) &= x_2 - \max_{f_1 \in F_1} u_2(f_1, h_2) \quad (h_2 \in H_2)
\end{aligned}$$

Notice that (2) implies $\delta(h_i) > 0$. Let $\delta_i = \min_{h_i \in H_i} \delta(h_i)$. Let $\varepsilon_j > 0$ $(j = 3 - i)$ satisfy

$$(1-\varepsilon_j)(x_i - \delta_i) + \varepsilon_j m_i^+ < (1-\varepsilon_j)x_i + \varepsilon_j m_i^-.$$

Let $s_i(a_i)$ denote the weight that s_i puts on a_i. Define $V_i(\varepsilon_i) = \{s_i \in V_i | \sum_{h_i \in H_i} s_i(h_i) < \varepsilon_i\}$. Then $V_1(\varepsilon_1) \times V_2(\varepsilon_2)$ is an open neighborhood of $R \cap \bar{R}$. We claim that this neighborhood is absorbed by $R \cap \bar{R}$.

Let $s_2 \in V_2(\varepsilon_2), h_1 \in H_1, f_1 \in F_1$. Now

$$\begin{aligned} u_1(h_1, s_2) &\leq \sum_{F_2} s_2(f_2)(x_1 - \delta(h_1)) + \sum_{H_2} s_2(h_2) m_1^+ \\ &< (1-\varepsilon_2)(x_1 - \delta_1) + \varepsilon_2 m_1^+ \\ &< (1-\varepsilon_2)x_1 + \varepsilon_2 m_1^- \\ &\leq \sum_{F_2} s_2(f_2) x_1 + \sum_{H_2} s_2(h_2) u_1(f_1, h_2) = u_1(f_1, s_2) \end{aligned}$$

Hence, there are no best replies against strategies in $V_2(\varepsilon_2)$ in H_1. Hence, the best replies are in F_1. Reversing the roles of the players then proves the claim. Hence, every persistent retract is contained in $\bar{R}$. □

Chapter 5

Signaling by Burning Money

5.1 Introduction

In the previous chapter we studied pre-play communication in games with symmetric and perfect information. Players can send messages to indicate which action they take in the game. The messages may serve as a coordination device. However, they do not reveal any new information. Signaling games are different. In these games some players have private information. The action taken by a privately informed player will typically depend on this information. Therefore, it may reveal (some of) this information to players who still have to take an action. These players will take that information into account when they make their decision.

There are many economic situations in which asymmetric information and signaling play a role. For example, an incumbent firm may have better information about demand and cost parameters than a potential entrant. Now it may happen, for instance, that a high price set by the incumbent, signals that demand is high, and therefore provokes entry. As another example, consider a worker who is privately informed about his ability and skill. He may try to signal his skill to a potential employer by choosing an appropriate education level. Of course, low skilled workers will want to hide their information, that is, they want to pool with the high skilled workers.

In this chapter we will only be concerned with the most basic form of a signaling game, namely the Sender-Receiver game. There are two players, the sender and the receiver. The sender has some private information (about his type) and sends a message to the receiver. The receiver then takes an action. Payoffs for both players depend on the type of the sender and the action taken by the receiver (and sometimes also on the message).

A drawback of the use of Sender-Receiver games is that they often give rise to a host of equilibria, giving the theory little predictive power. This is often the case in games with "cheap talk". Here the sender can send any message from some (finite) set of available messages without bearing any direct cost. (The payoffs are indirectly dependent on the message because of the influence of the message on the decision of the receiver.) The introduction of refinements can reduce the set of equilibrium outcomes. Blume (1994) shows that persistence (Kalai and Samet (1984)) rules out pooling outcomes in games that have strict separating equilibria, if the message space is "small". Consider for example a Sender-Receiver game in which the two types of sender are equally likely. Payoffs are indicated in Figure 5.1: Each cell corresponds to one type-action pair. The first entry is the payoff for the sender, the second entry is the payoff for the receiver.

	a_1	a_2	a_3
t_1	1,3	0,0	2,2
t_2	0,0	1,3	2,2

Figure 5.1.

If there are only two (costless) messages available, then the pooling equilibrium (i.e., the equilibrium in which both types of sender choose the same message), which is attractive to the sender, is not persistent.

The results of Blume (1994) are not robust to an extension of the message space. This is caused by the fact that if there are unused messages, the receiver is indifferent between a lot of best replies. This implies that the sender can be indifferent between using some messages because they can lead to the same action and they are all costless. Indifferences cause "drift", i.e. the receiver may switch between different best replies and the sender may switch between messages. This amounts to very large persistent retracts that contain both pooling and separating equilibria. Hence, the predictive power is very low.

Blume (1993a) introduced the notion of perturbed message persistency. Roughly speaking, he considers persistent equilibria of the signaling game in which the strategy of the sender is perturbed. This means that involuntarily used messages have some residual meaning, and the receiver is no longer indifferent. One of Blume's results is that, if there exists an equilibrium in which all types of sender get their preferred action, then any perturbed message persistent retract consists of such an equilibrium. In order to obtain this result he assumes that the perturbation is a so called effective language. This means that there are some nominal costs for certain messages, and that the message space is very large.

In this chapter we assume that messages are costly, and more importantly that they all have different costs. These differences in the costs of the messages let the indifference on the side of the sender disappear. An advantage of our approach is that we only need that the number of available messages exceeds the number of types. The results of this chapter are therefore robust to an extension of the message space.

The main result of this chapter is that, when there exists an equilibrium in which all types of sender get their preferred action, then every strategy profile in a persistent retract yields all types of sender their preferred action. This is also true if "persistent" is replaced by "curb" or "curb*" (Basu and Weibull (1991)). For the definitions and some properties of these solution concepts the reader is referred to Chapter 2.

5.2 Sender-Receiver games

Nature draws the type of sender: $t \in \mathcal{T} = \{t_1, \ldots, t_l\}$ with probability $p(t) > 0$. The sender learns his type and then sends a message $m \in \mathcal{M} = \{m_1, \ldots, m_M\}$ to the receiver. Observing the message sent but not knowing the type of the sender, the receiver chooses an action $a \in \mathcal{A} = \{a_1, \ldots, a_n\}$. We will consider only the normal form of this signaling game. Hence, the set of pure strategies for the sender is

$$\mathcal{F} = \{f : \mathcal{T} \to \mathcal{M}\},$$

and the set of pure strategies for the receiver is

$$\mathcal{G} = \{g : \mathcal{M} \to \mathcal{A}\}.$$

The payoff of a sender of type t that sends message m and gets action a is denoted by $u(t,a) - c(m)$. The payoff for the receiver is then $v(t,a)$. The total payoffs for the sender and receiver are, when the strategy combination (f, g) is used, respectively

$$\begin{aligned} U(f,g) &= \sum_{t \in \mathcal{T}} p(t)(u(t, g(f(t))) - c(f(t))) \\ V(f,g) &= \sum_{t \in \mathcal{T}} p(t) v(t, g(f(t))) \end{aligned}$$

We assume that every type of sender t has a unique preferred action a_t^*, i.e. $\{a_t^*\} = \arg\max_a u(t,a)$. Notice that different types may have the same preferred action. Without loss of generality we write $\{a_t^* | t \in \mathcal{T}\} = \{a_1, \ldots, a_k\}$ $(k \leq |\mathcal{T}|)$.

We assume that the number of messages exceeds the number of types, i.e. $|\mathcal{M}| > |\mathcal{T}|$. Moreover, it is assumed that different messages have different costs, and that there are enough cheap messages. Formally, we assume that $c(m_1) = 0$ and $c(m_i) < c(m_{i+1})$. Furthermore, for all $t \in \mathcal{T}$ and all $a \neq a_t^*$

$$u(t, a_t^*) - c(m_{|\mathcal{T}|+1}) > u(t, a).$$

This assumption implies that if the sender has the possibility to induce his preferred action by sending a relatively cheap message, he will do so.

Before we go further we need some more notation.

$$f(\mathcal{T}) = \{f(t)|t \in \mathcal{T}\} \quad (\text{all } f \in \mathcal{F})$$

$$f^{-1}(m) = \{t \in \mathcal{T}|f(t) = m\} \quad (\text{all } f \in \mathcal{F}, m \in \mathcal{M})$$

Now we come to a major assumption.

Assumption 5.1 *There exists a pure strategy combination (f, g) satisfying*

(i) $g(f(t)) = a_t^*$ *for all* $t \in \mathcal{T}$

(ii) for all $m \in f(\mathcal{T})$, $\{g(m)\} = \arg\max_a \sum_{t \in f^{-1}(m)} p(t)v(t, a)$

This assumption tells us that the cheap talk variant of the game has a pure Bayesian Nash equilibrium in which all types of sender get their preferred action and in which the receiver's best replies after equilibrium messages are unique.

This does not exclude the possibility that there might exist equilibria that are better for the receiver. Consider again the example we studied in the introduction. If types are equally likely then the pooling equilibrium yields all types of sender the preferred action a_3 and the receiver a payoff of 2, but the separating equilibria yield the receiver 3.

Theorem 5.1 *Every strategy profile that is contained in a persistent, curb or curb* retract of the costly signaling game is an equilibrium and yields all types of sender the preferred action. Moreover, the messages used in such equilibria are the k cheapest ones.*

Proof. Note that, as long as $|\mathcal{A}| > 1$, the sender has no pure equivalent strategies. Therefore, we know that the extreme points of any persistent retract are pure strategies.

We introduce some notation.

$$P^i = \{t|a_t^* = a_i\}$$

$$P(t) = \{t'|a_{t'}^* = a_t^*\}$$

$\mathcal{P}_k$ is the set of permutations of $\{1, \ldots, k\}$

$\mathcal{F}^* = \{f \in \mathcal{F} | f(\mathcal{T}) = \{m_1, \ldots, m_k\}$ and $\exists\, \tau \in \mathcal{P}_k$ s.t. $f^{-1}(m_i) = P^{\tau(i)}\}$

Hence, $\mathcal{F}^*$ is the set of pure strategies for the sender that partitions the types according to their preferred action, using only the k cheapest messages.

The proof consists of three steps. The first step is quite complicated. Therefore, we first give the proof under the stronger assumptions that $|\mathcal{M}| \geq 2|\mathcal{T}|$ and that, for all $t \in \mathcal{T}$ and all $a \neq a_t^*$,

$$u(t, a_t^*) - c(m_{2|\mathcal{T}|}) > u(t, a).$$

Step 1. Let R be a persistent, curb* or curb retract. Let $f \in R_1$ be a pure strategy. Let $m_{i_1}, \ldots, m_{i_k}$ be the k cheapest unused messages in f. There exists $g \in B(f) \cap R_2$ with $g(m_{i_j}) = a_j$ $(j = 1, \ldots, k)$. Then for every $f' \in B(g)$ we have $g(f'(t)) = a_t^*$, for all t. Moreover, all types $t' \in P(t)$ choose the same message (namely the cheapest message that induces a_t^*). Hence, $g \in B(f')$.

Step 2. Let R be a persistent, curb* or curb retract. Let $(f, g) \in R$ be an equilibrium in which all types of sender get their preferred action and in which all types $t' \in P(t)$ use the same message, for all t. Then there are k messages used, but it need not be the k cheapest ones. Let $m_{j_1}, \ldots, m_{j_r}$ be the used messages that are at least as expensive as m_{k+1}. Then there are also r unused messages that are at least as cheap as m_k: $m_{i_1}, \ldots, m_{i_r}$. There must be $g' \in R_2$ such that $g'(m) = g(m)$ if $m \in f(\mathcal{T})$ and $g'(m_{i_s}) = g(m_{j_s})$ $(1 \leq s \leq r)$. Now $B(g') \subset \mathcal{F}^*$.

Step 3. Let $f \in \mathcal{F}^*$. Then $\{f\} \times \mathrm{co}(B(f))$ is absorbing, has the curb* property and is a curb retract. Hence, every persistent, curb* or curb retract that contains f is contained in the above retract.

Step 1 showed us that any persistent, curb* or curb retract contains an equilibrium in which all types of sender get their preferred action. Step 2 shows then that it must also contain an equilibrium in which the sender plays a strategy from the set $\mathcal{F}^*$. By step 3 we know that all persistent, curb* or curb retracts that contain $f \in \mathcal{F}^*$ are contained in one of the retracts described there. Every strategy profile in the retracts described there is an equilibrium and gives all types of sender the preferred action and uses only the cheapest k messages. This completes the proof under the stronger assumption on the number of messages.

Now we continue without making this stronger assumption. The proof of step 1 given before, is now no longer correct, because we cannot be sure that there are k unused messages. Steps 2 and 3, however, remain valid. We only need to adjust the proof of step 1. The basic idea of the proof is to separate the types step by step.

First, we introduce some more notation.

$$\mathcal{F}^i = \{f \in \mathcal{F} | \forall t, t' \in P^i \ \forall t'' \notin P^i \ \ f(t) = f(t') \neq f(t'')\}$$

$$\mathcal{F}_S = \{f \in \mathcal{F} | t, t' \in S \ \Rightarrow \ [\, f(t) = f(t') \ \Leftrightarrow \ t' \in P(t)\,]\} \quad (S \subset \mathcal{T})$$

$\mathcal{F}^i$ is the set of strategies for the sender that separates the types in P^i from other types. Note that $\mathcal{F}_\mathcal{T} = \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^k$. We need to show that $\mathcal{F}_\mathcal{T} \cap R_1 \neq \emptyset$.

Let $f \in R_1 \cap \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^s$ $(0 \leq s < k)$. Since there is at least one unused message, there exists $g \in B(f) \cap R_2$ such that for all $f' \in B(g)$

$$\text{for all } t \in \cup_1^{s+1} P^i, \quad g(f'(t)) = a_t^*.$$

Hence, for all $f' \in B(g)$ we have $f' \in \mathcal{F}_{\cup_1^{s+1} P^i}$. (And at least one such f' must be in R_1.) Let m_{i_j} denote the message sent (according to this f') by types in P^j $(1 \leq j \leq s+1)$.

Now we can have two different cases. Either $f' \in \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^{s+1}$ or

$$\exists \tilde{t} \notin \cup_1^{s+1} P^i \text{ s.t. } f'(\tilde{t}) \in \{f'(t) | t \in \cup_1^{s+1} P^i\}. \tag{5.2.1}$$

Consider the latter case. Let

$$z_j = \begin{cases} 1 & \text{if } \exists \tilde{t} \notin \cup_1^{s+1} P^i \text{ s.t. } f'(\tilde{t}) \in \{f'(t) | t \in P^j\} \\ 0 & \text{otherwise} \end{cases}$$

W.l.o.g. $z_1 = \ldots = z_y = 1$ and $z_{y+1} = \ldots = z_{s+1} = 0$. Then there are at least $y+1$ unused messages in f', say $\tilde{m}_1, \ldots, \tilde{m}_{y+1}$. Then there exists $g' \in B(f') \cap R_2$ such that

$$g'(m_{i_j}) = g(m_{i_j}) \text{ if } y+1 \leq j \leq s+1$$

$$g'(\tilde{m}_j) = g(m_{i_j}) \text{ if } 1 \leq j \leq y$$

$$g'(\tilde{m}_{y+1}) = a_{\tilde{t}}^* \text{ where } \tilde{t} \text{ is as in (5.2.1)}$$

Now we have for all $f'' \in B(g')$ that $f'' \in \mathcal{F}_{\cup_1^{s+1} P^i \cup P(\tilde{t})}$.

Again either $f'' \in \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^{s+1} \cap \mathcal{F}^x$ (where x is such that $P^x = P(\tilde{t})$) or there exists $t' \notin \cup_1^{s+1} P^i \cup P^x$ such that

$$f''(t') \in \{f''(t) | t \in \cup_1^{s+1} P^i \cup P^x\}.$$

(Compare with (5.2.1).)

Repeating the same argument at most $|\mathcal{T}|$ times gives us

$$\mathcal{F}^1 \cap \ldots \cap \mathcal{F}^s \cap R_1 \neq \emptyset \quad \Rightarrow \quad \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^{s+1} \cap R_1 \neq \emptyset.$$

Repeating this argument at most k times gives us

$$R_1 \cap \mathcal{F}_{\mathcal{T}} = R_1 \cap \mathcal{F}^1 \cap \ldots \cap \mathcal{F}^k \neq \emptyset.$$

Now step 2 and 3 can be applied and the proof is completed. □

Note that the bound $|\mathcal{T}| > |\mathcal{M}|$ is tight: We already saw that the separating equilibrium of the cheap talk game from Figure 1 forms a persistent retract if there are only two messages. This remains to hold when the messages are costly.

One might think that the theorem can be generalized in the following sense. If there exists an equilibrium in which all types $t \in \mathcal{T}' \subset \mathcal{T}$ of sender get their preferred action, then in every equilibrium that is contained in a persistent, curb or curb* retract all types in $\mathcal{T}'$ get their preferred action. This is however not true in general. The following example shows a game in which for every type there is an equilibrium which yields that type the preferred action. But there does not exist an equilibrium in which all types get the preferred action.

	a_1	a_2	a_3	a_4
t_1	5,3	1,4	0,0	3,3
t_2	5,3	0,0	1,4	0,0
t_3	0,0	0,0	1,4	5,3

Figure 5.2.

All types are equally likely. The equilibria in which t_1 and t_2 pool, yield these types their preferred action a_1. The equilibria in which t_1 and t_3 pool, yield t_3 the preferred action a_4. But there does not exist an equilibrium in which all types get their preferred action.

5.3 Related literature and concluding remarks

Cheap talk games often give rise to a multitude of equilibria because unused messages cause indifferences on the side of the sender and the receiver. In this chapter we got rid of the indifference on the side of the sender by assuming that messages are costly. Blume, Kim and Sobel (1993) made the same assumption. They used the notion of an

EES set (Swinkels (1992), see also Chapter 2) and obtained a slightly weaker result: EES sets exist and consist of efficient equilibria (less of signaling costs) *if* the interests of the sender and the receiver are aligned (Propositions 3 and 4). In case the interests are not aligned, they show that there exists an EES set in which the sender gets the preferred action (Proposition 5). However, they also provide an example that shows that there may exist other EES sets in which the sender does not get the preferred action (Figure 3, p. 560).

Blume (1993a) solved the indifference on the side of the receiver by introducing perturbations, which give residual meaning to unused messages. Another possibility to solve the indifferences was introduced by Wärneryd (1993). He assumes that the receiver has a preference for using less complex strategies.

In this chapter we applied persistency. In Chapter 3 we presented a learning process that results in the play of persistent strategies. However, it was assumed there that players observe the strategies used in the past. Probably it makes more sense to assume that players only observe the outcomes and not the full descriptive strategies, if the game is in fact an extensive form game. However, the learning process could be adjusted to produce the same result if players only observe outcomes in so called single curb outcome games, i.e. games with the property that all strategies in the same curb retract yield the same outcome. There was a problem with games with moves of Nature: It might happen that in a bounded history certain types are never chosen by Nature. It is interesting to note that Canning (1993) considers a Sender-Receiver game with costly messages, very similar to the model of this chapter. He presents a learning process that is similar to the one of Chapter 3. By assumption all types are always present. Under the assumption that players draw a random belief for those messages that are not observed in the recent history, Canning (1993) shows that in common interest signaling games with probability one the efficient (fully separating) equilibrium is obtained.

Chapter 6

Commitment Robust Equilibria and Endogenous Timing

6.1 Introduction

One of the most important ideas in game theory is the value of commitment, the idea that it may be advantageous to constrain one's own behavior in order to induce others to behave in a way that is favorable to oneself. Schelling's (1960) classic *The Strategy of Conflict* is filled with examples that illustrate this idea that it might pay to reduce one's flexibility, that it may be optimal to burn the bridges behind oneself. The simplest commitment possibility that Schelling discusses (and what he calls the "pure unconditional commitment") is equivalent to obtaining the first move: to preempt one's rivals by choosing and communicating an action before they do. In the economics literature, this first-mover advantage has been known at least since Von Stackelberg (1934) pointed out that in a quantity setting duopoly game, the leader has higher profits than the follower and than a Cournot competitor. Various authors have argued that the fact that each duopolist has an incentive to move earlier than his opponent makes the Cournot equilibrium somewhat suspect. Of course, if the duopolists are indeed forced to move simultaneously (as the standard game model of the duopoly situation assumes), then there is nothing wrong with the Cournot equilibrium, but one may wonder whether in real situations the rules are indeed that rigid as to prevent commitments from being made.

The above observation naturally leads to the more general question of which Nash equilibria are still viable when players have the opportunity to commit themselves. This question has been addressed recently in Rosenthal (1991). Rosenthal defines an equilibrium s of a 2-person game g to be commitment robust if s is also a subgame perfect equilibrium outcome of each of the two games where one of the players moves first. He argues that "failure of commitment robustness ought to signal that there is a sense in which the equilibrium is questionable if in the intended application the rules can be modified in the appropriate way sufficiently inexpensively by the player who stands to gain". However, this argument is not entirely compelling. Rosenthal compares the simultaneous move game with the two perfect information sequential move games in which the leader is exogenously specified, but it is not clear why the latter games are relevant to the study of the former situation. The original problem derives from the possibility that each player might or might not choose to move before the other. Both players simultaneously have the possibility to commit themselves, no player can unilaterally change the rules of the game, hence, one would like to see the commitments arise endogenously. Rosenthal seems well aware of this problem with the definition that he proposes. He writes: "In defining commitment-robustness, one might require consideration of more than just the alternative games G_I and G_{II}; after all there could be opportunities for both players to invest in commitment possibilities. It seems best, therefore, to think of the defining conditions here as being in the nature of necessary conditions."

The question we would like to raise is whether Rosenthal's definition indeed gives necessary conditions. Specifically, could it not be that, even though each player could profit from moving first, no player dares to commit since he fears that the opponent might commit simultaneously? Hence, could it not be that, as a consequence, the players end up in the Nash equilibrium after all? Certainly, the latter seems a real possibility in cases where committing to an action is risky, as it is, for example, in the quantity setting duopoly game. To address these issues we make use of the game of action commitment that was proposed in Hamilton and Slutsky (1990). The rules are as follows: There are two periods and each player has to move in exactly one of these periods. Choices are simultaneous, but, if one player chooses to move early (i.e. to commit) while the other moves late, the latter is informed about the former's choice before making his decision. Hence, in this game, each player can commit, no player has the sole privilege of being able to do so. We investigate which equilibria of the original game can arise as "sensible" equilibrium outcomes of the action commitment game. Are indeed only those equilibria viable at which no player has an incentive to move first? Is commitment robustness, as defined by Rosenthal, a necessary condition for equilibrium outcomes to survive when the sequencing of the moves is endogenous?

A first important result is that a mixed strategy equilibrium indeed is viable only if no player has an incentive to move first at this equilibrium (Theorem 6.2). Only in this case does the action commitment game have a subgame perfect equilibrium that produces this outcome. The intuition is quite straightforward: The actions actually resulting from the players' mixed strategies need not be in equilibrium, ex post players have an incentive to deviate. Hence, each player will have an incentive to wait, thereby guaranteeing that he is best responding no matter which action the opponent is actually choosing. But if a player waits, the opponent frequently will prefer to commit to his Stackelberg leader strategy.

For pure strategy equilibria, the situation is much different and it turns out that these are not as easily eliminated: An immediate commitment of each player to this equilibrium is part of a subgame perfect equilibrium of the 2-stage action commitment game (Theorem 6.3). Furthermore, such an immediate commitment constitutes a perfect equilibrium in the normal form of the action commitment game (Theorem 6.4). Intuitively, if each player expects an unattractive outcome in case the timing game reaches the second stage, then it is optimal for each player to commit to the pure equilibrium immediately if he expects his opponent to do the same. Hence, requiring perfectness does not allow one to conclude that only pure equilibria in which no player has an incentive to move first are viable when the timing is endogenous. Consequently Rosenthal's conclusion appears premature.

To see whether Rosenthal's intuition can be given some foundation we then turn to more refined equilibrium notions. The literature offers two types of refinements, the distinction being whether an evolutionary or an eductive interpretation of equilibria is adopted. An example that we discuss extensively in the next section shows that the eductive perspective does not allow one to conclude that only commitment robust equilibria are viable when the timing of the moves is endogenous: Also other pure equilibria may result from proper (Myerson (1978)) and even stable (Kohlberg and Mertens (1986)) equilibria of the action commitment game.

The main result of this chapter, however, shows that Rosenthal's intuition can be supported by a variety of set-valued solution concepts that have an evolutionary flavor. These concepts seem to correspond more closely to the interpretation of an equilibrium as a fixed point of an unspecified dynamic process, than to the interpretation of an equilibrium as a self-enforcing agreement. Specifically, our main result shows that the concepts of persistent equilibria (Kalai and Samet (1984)) and curb*-equilibria (Basu and Weibull (1991)) force players to coordinate on the commitment robust equilibrium whenever this is unique and pure (Theorem 6.6). Hence, if we accept curb* or persistent equilibria as the relevant solution concept, the results in this chapter allow us to identify two classes of games for which we can unambiguously determine the outcome if the order of the moves is endogenous, viz. zero-sum games and games with common interest. While the result concerning zero-sum games is not surprising, it is remarkable that one has to turn to very restrictive equilibrium notions to "justify" playing the Pareto-efficient equilibrium in common interest games.

The remainder of this chapter is organized as follows. In Section 6.2 we give an example of a coordination game to show that the intuition, that only commitment robust equilibria are viable when the timing is endogenous, might be misleading. In Section 6.3 we introduce notation and give a definition of commitment robust equilibria that differs slightly from that of Rosenthal. We compare the two concepts and derive characterizations of them. Section 6.3 also formally introduces the 2-stage action commitment game. In Section 6.4 we show that a mixed strategy equilibrium is viable with endogenous timing only if it yields each player at least his Stackelberg leader payoff. In contrast it is shown in Section 6.5 that requiring perfectness in the timing game does not eliminate pure equilibria of the underlying game. In this section we also discuss the related work of Hamilton and Slutsky (1990). Section 6.6 investigates whether players will automatically coordinate on a commitment robust equilibrium when the order of the moves is endogenous. Section 6.7 concludes and discusses some related work.

6.2 A coordination game

In this section we discuss a simple example to show that the intuition that, when the timing of the moves is endogenous, only equilibria in which no player has an incentive to move first are viable, is not completely reliable. We show that one needs to employ quite restrictive solution concepts in order to arrive at this conclusion. Specifically, the following example demonstrates that even if a game has a strict equilibrium that is also the unique Stackelberg equilibrium outcome in each of the two games where one of the players is forced to move first, it is by no means obvious that players will necessarily coordinate on this equilibrium when the order of the moves is determined endogenously.

Consider the common interest coordination game g given in Figure 6.1a. When players have to move simultaneously, there are three equilibria: (T, L), (B, R) and a mixed equilibrium that yields each player the payoff $\frac{2}{3}$. When one of the players is forced to move first, with this player's choice being revealed to the other, the subgame perfect outcome is (T, L), irrespective of which player moves first. According to Rosenthal's definition only the equilibrium (T, L) is commitment robust.

	L		R	
T	2	2	0	0
B	0	0	1	1

Figure 6.1a: A coordination game.

	L^1		L^2		R^2		R^1	
T^1	2	2	2	2	2	2	0	0
T^2	2	2	2	2	0	0	1	1
B^2	2	2	0	0	1	1	1	1
B^1	0	0	1	1	1	1	1	1

Figure 6.1b: The (reduced) normal form of the 2-period game of endogenous timing associated with the game from Figure 6.1a.

Let us investigate whether this equilibrium is the unique "sensible" outcome of the 2-stage game of action commitment, described in the introduction, in which the order of the moves is endogenously determined. In this action-commitment game both players have 10 pure strategies. (A player has two possibilities to commit himself; if he does not commit he might be in three different information sets in period 2 and in each of these he can choose between two possible actions.) Figure 6.1b displays a reduced form of this normal form: It only lists those strategies in which a player plays a best response in the second period whenever the opponent has unilaterally committed himself in the first period. Since this reduction makes it easier to see the intuition for our results, our discussion in this section will be based on it. Note, however, that our formal results are based on the full, unreduced, game. In Figure 6.1b, X^1 denotes the pure strategy "commit to X in period 1", while X^2 denotes "wait till period 2, play the unique best response if the opponent has already moved in period 1 and, otherwise, play X." The question we want to address is whether "2,2" (i.e. the commitment robust equilibrium) is the unique "sensible" outcome in the game of Figure 6.1b. Alternatively, can one perhaps also give good arguments in favor of "1,1", or in favor of the mixed equilibrium?

Inspection shows that the game from Figure 6.1b has three Nash equilibrium outcomes: There is a connected set of equilibria with payoff (2,2), there is also a connected set of equilibria with payoff (1,1) and there is a completely mixed equilibrium in which each player plays (1,2,4,8)/15. The latter yields each player the payoff 14/15. Hence, we see that the endogenous sequencing of the moves eliminates the mixed strategy equilibrium. The intuition is simple: If my opponent commits himself to the mixed equilibrium in the first stage, I have an incentive to wait since in that case I guarantee that we either coordinate on (T, L) or (B, R), i.e. I prevent a disequilibrium outcome. However, if I wait, my opponent does better by committing to his Stackelberg leader strategy.

Having eliminated the mixed equilibrium from Figure 6.1a, let us now turn to the Pareto dominated pure strategy equilibrium "1,1". A glance at Figure 6.1b shows that this is not as easily eliminated. Namely, B^1 and R^1 are undominated strategies in Figure 6.1b so that (B^1, R^1) is a perfect equilibrium. Even stronger, (B^1, R^1) is a proper equilibrium (Myerson (1978)): For any $\varepsilon > 0$ that is sufficiently small the strategy pair in which each player plays the completely mixed strategy $(\varepsilon^3, \varepsilon^2, \varepsilon, 1 - \varepsilon - \varepsilon^2 - \varepsilon^3)$ is a 2ε-proper equilibrium in Figure 6.1b. In fact, one may show that, in the game of Figure 6.1b, the connected set of equilibria with payoff "1,1" contains a stable set as defined in Kohlberg and Mertens (1986). Intuitively, if players expect mistakes to occur with a relatively large probability in the second period, or if they believe that the unattractive mixed strategy equilibrium would be played in case the second period would be reached, then each player has a strong incentive to commit to the equilibrium "1,1" in the first period if he expects his opponent to do the same. Hence, it seems that Rosenthal's (1991) conclusion that only (T, L) makes sense in the game of Figure 6.1a when aspects of commitment are taken into account was premature.

The concept of Nash equilibrium allows two interpretations. The first is as a necessary requirement for a self-enforcing agreement and the concepts of perfectness, properness and stability all correspond to this interpretation. Hence, from the preceding paragraph we may conclude that an agreement on "1,1" is self-enforcing even when the timing of the moves is endogenous. An alternative interpretation of a Nash equilibrium is as a fixed point of an (unspecified) dynamic evolutionary or learning process. Several refinements have been proposed that correspond to this interpretation and it turns out that some of these allow the conclusion that players will coordinate on "2,2" when the timing of the moves is endogenous. Specifically, the concepts of persistent equilibria (Kalai and Samet (1984)) and curb*-equilibria (Basu and Weibull (1991)) force players to coordinate on (T, L). Let us illustrate this result for curb*-equilibria, i.e. for equilibria that belong to minimal sets of mixed strategy pairs that are closed with respect to taking undominated best responses.[1] The game of Figure 6.1b has two sets that are closed under (undominated) best responses, viz. the entire strategy set and the set of strategies in

[1] Since we here consider the reduced game, in this section this result holds also for curb-equilibria, i.e. equilibria that belong to minimal sets of mixed strategies that are closed with respect to taking best responses.

which zero probability is assigned to B^1 and R^1. Obviously, only the latter set is minimal. Furthermore, each equilibrium in this set induces the outcome "2,2", hence, each curb* equilibrium of the game of Figure 6.1b produces the commitment robust equilibrium of Figure 6.1a. In Section 6.6 we show that the above argument generalizes to any game that admits a unique and pure commitment robust equilibrium. In particular, for each common interest game we, hence, have a "justification" for playing the Pareto-dominant equilibrium.

6.3 Commitment robust equilibria

We start our analysis from a given finite 2-person normal form game $g = (A_1, A_2, u_1, u_2)$. To avoid trivial cases we assume throughout this chapter that $|A_i| \geq 2$. We write $S_i = \Delta(A_i)$ for the set of mixed strategies of player i in g and we denote a generic mixed strategy by s_i. We write $C(s_i) = \{a_i \in A_i : \; s_i(a_i) > 0\}$. The set of mixed strategy pairs is $S = S_1 \times S_2$, and, for $s \in S$, $C(s) = C(s_1) \times C(s_2)$. Throughout this chapter we assume the mild regularity condition that different outcomes are associated with different payoffs, i.e.

$$\text{if } a \neq a', \text{then } u_i(a) \neq u_i(a') \quad (i \in \{1,2\}) \tag{6.3.1}$$

Obviously, (6.3.1) is satisfied for generic games. This condition implies that the best reply against any pure strategy is unique. We denote the best reply of player j against a_i by $b(a_i)$ and write

$$u_i(a_i) = u_i(a_i, b(a_i)) \tag{6.3.2}$$

and

$$\bar{u}_i \;\; = \;\; \max_{a_i \in A_i} u_i(a_i). \tag{6.3.3}$$

Hence, $u_i(a_i)$ is player i's payoff when he commits to a_i and $\bar{u}_i$ is this player's payoff when he acts as the Stackelberg leader. Note that (6.3.1) implies that there is a unique strategy attaining the maximum in (6.3.3). We say that a player has an incentive to move first if he would be better off if he would be allowed to act as a Stackelberg leader.

If no player prefers being a leader to playing simultaneously, the equilibrium is said to be commitment robust. Formally:

Definition 6.1 *(i) Player i has an incentive to move first at s if $u_i(s) < \bar{u}_i$.*

(ii) An equilibrium is commitment robust (is a CRE) if no player has an incentive to move first at s.

Note that any pure strategy Nash equilibrium s satisfies $u_i(s) \leq \bar{u}_i$ for $i = 1,2$. Hence, if a pure equilibrium s is a CRE, then $u_i(s) = \bar{u}_i$ for $i = 1,2$ and s is a subgame perfect equilibrium outcome in each of the two games where one of the players is forced to move first. Conversely, if s is pure and s is a subgame perfect equilibrium outcome of each of these two games, then s is a Nash equilibrium of g and no player has an incentive to move first at s. Obviously, any such equilibrium must Pareto-dominate any other pure Nash equilibrium and we have proved

Corollary 6.1 *A pure strategy equilibrium s is a CRE if and only if s is an SPE outcome of the two games in which one of the players is forced to move first. A pure CRE Pareto-dominates any other pure Nash equilibrium. Hence, two pure CRE are payoff equivalent, and, therefore, each game has at most one pure CRE.*

It is easy to construct games without any CRE, take the Battle of the Sexes, for example. On the other hand, the set of games with a pure CRE is certainly not of measure zero. All games that are close to the game from Figure 6.1a have (T, L) as a pure CRE. Two classes of games that do admit CRE are the strictly competitive games (Friedman (1983)) and the common interest games (Aumann and Sorin (1989)). The game g is said to be *strictly competitive* if for all s, s' and $i \neq j$, if $u_i(s) \geq u_i(s')$, then $u_j(s) \leq u_j(s')$. The game g is a game with *common interest* if there exists s such that $u_i(s) \geq u_i(s')$ for all i and s'.

Lemma 6.1

a) If g is strictly competitive and s is a Nash equilibrium of g, then s is a CRE.

b) If g has common interest and s is the Pareto dominant Nash equilibrium of g, then s is a CRE.

Proof. The proof of *b*) is trivial, so we only prove *a*). Assume s is an equilibrium of a strictly competitive game g and $u_1(s) < \bar{u}_1$. Let $\bar{u}_1 = u_1(\bar{a}_1) = u_1(\bar{a}_1, \bar{a}_2)$. Then $u_2(\bar{a}_1, a_2) \leq u_2(\bar{a}_1, \bar{a}_2)$ for all a_2, hence, $u_1(\bar{a}_1, a_2) \geq u_1(\bar{a}_1, \bar{a}_2) > u_1(s)$ for all a_2. But then by playing $\bar{a}_1$ player 1 could guarantee himself more than the equilibrium payoff $u_1(s)$, which is impossible. □

Corollary 6.1 is no longer correct for mixed equilibria. For example, the zero-sum game of Matching Pennies has a unique (symmetric) mixed strategy equilibrium. No player has an incentive to move first at this equilibrium: each player is sure to lose the game if he is forced to disclose his action before the opponent. However, the game in which player i is forced to move first has no SPE with the same outcome as the mixed equilibrium of Matching Pennies. Furthermore, a pure CRE may be Pareto dominated by a mixed equilibrium: Add to Matching Pennies a third strategy for each player such that if both play this strategy both lose half a penny, but both players lose one penny if only one player chooses this strategy. In each Stackelberg outcome, both players choose the third strategy, but the resulting equilibrium is Pareto dominated by the original mixed equilibrium of Matching Pennies.

The above problems associated with mixed strategies can be avoided by considering the two games where one player is allowed, but not forced, to move first. Hence, in these games the timing of one player's decision is determined endogenously. Formally, in round 1 player i chooses between to commit to an action a_i or not. If he commits himself, then the commitment a_i is revealed to player j. If not, player j is informed that player i did not commit himself and players choose their actions simultaneously. The reader can now easily verify that s is a CRE if and only if each of these two games has a subgame perfect equilibrium with outcome s.

Corollary 6.2 *An equilibrium s is a CRE if and only if s is an SPE outcome in each of the two games in which one player is allowed to move first.*

The definition of commitment robustness given above differs from that given in Rosenthal (1991). The difference concerns the treatment of mixed strategies. Rosenthal defines an equilibrium s to be commitment robust if s is an SPE outcome of each of the two games in which one of the players moves first by committing to a mixed strategy. Specifically, Rosenthal assumes that the mixed strategy that the player commits to can be communicated. Hence, he works with what Schelling (1960, p. 185) calls "fractional commitments" and Schelling already points out that these may be more efficient than pure ones. (Compare our discussion of the extended matching pennies game in which a mixed strategy equilibrium Pareto dominates the pure strategy equilibrium.) Of course, having the opportunity to commit to a mixed action can never be worse than having the opportunity to commit to a pure action and, hence, any equilibrium that is commitment robust according to Rosenthal (which will be called an RCRE) is also a CRE. As we have seen, a CRE need not be an RCRE. However, the reader may easily verify that for each of the two classes of games considered in Lemma 6.1 any CRE is also an RCRE.

As Corollary 6.2 shows, there are three differences between our concept and that of Rosenthal. First, although players can commit to mixed actions in both settings, only Rosenthal allows them to communicate these mixtures. In our case, the actual outcome of the randomization has to be communicated. Secondly, we allow the players to commit themselves, but we do not force them. In Rosenthal's setup players are forced to commit themselves. Another difference between the two concepts is that our concept is characterized by auxiliary games, while Rosenthal uses such games directly in the definition of his concept. The following theorem shows that, for games satisfying a regularity condition that is slightly different from (6.3.1), RCRE could have been similarly defined directly by means of a pair of inequalities. The regularity condition in question has first been introduced in Lemke and Howson (1964) and is also satisfied for generic games. For $s_j \in S_j$ write $B(s_j)$ for the set of pure best replies of player i against s_j and define the matrix u_{is_j} as the restriction of u_i to $B(s_j) \times C(s_j)$. The regularity condition that Lemke and Howson impose is

$$\text{rank } (u_{is_j}) \geq |B(s_j)| \text{ for all } j \in \{1,2\} \text{ and all } s_j \in S_j. \tag{6.3.4}$$

Condition (6.3.4) states that there are no linear dependencies between the rows and the columns of the payoff matrices. It implies that the equilibria are isolated and that the number of equilibria is odd.

Theorem 6.1 *For a game g and a player i define u_i^+ by means of*

$$u_i^+ = \max_{s_i \in S_i} \max_{a_j \in B(s_i)} u_i(s_i, a_j).$$

Let s^* be a Nash equilibrium of g. If $u_i(s^*) \geq u_i^+$ for $i = 1,2$, then s^* is an RCRE. If g satisfies (6.3.4), then s^* is an RCRE if and only if $u_i(s^*) \geq u_i^+$ for $i = 1,2$.

Proof. See Appendix. □

If g does not satisfy (6.3.4), then an RCRE s need not satisfy the inequalities $u_i(s) \geq u_i^+$ $(i = 1,2)$. The game from Figure 6.2 provides an example. (The game does not satisfy (6.3.1) but the payoffs may be slightly perturbed to satisfy this condition. However, care should be taken that the third pure strategy of player 2 remains a best response to the equilibrium strategy of player 1.) In this game, the equilibrium s in which both players choose each of the first two pure strategies with probability 1/2 is an RCRE. Player

1's equilibrium payoff is 1 while $u_1^+ = 3$, hence, $u_1(s) < u_1^+$. We note that a Nash equilibrium s satisfying $u_i(s) = u_i^+$ $(i = 1, 2)$ is called a Stackelberg equilibrium (SE) in d'Aspremont and Gérard-Varet (1980). Hence, we have that each SE is an RCRE and that each RCRE is a CRE and that neither of the converses holds. We leave further discussion to Section 6.7.

	L		C		R	
T	2	0	0	2	3	1
B	0	2	2	0	3	1

Figure 6.2.

In comparing our concept of commitment robustness to that of Rosenthal, we believe that the setting corresponding to our definition is more natural, however, we do not want to be dogmatic: In our main result, CRE may be replaced by RCRE. Perhaps more importantly, as we already argued in the introduction, both concepts are somewhat unsatisfactory since they are based on games with asymmetric commitment possibilities. We would want to give both players the possibility to commit themselves so that commitments could arise endogenously. Hence, we want to investigate which equilibria of the original game are still viable when the order of the moves is endogenous. To formally address this issue, we use the 2-stage game of action commitment that has been introduced in Hamilton and Slutsky (1990). In this game, which will be referred to as γ^2, each player can choose between committing to an action in period 1 or to wait till period 2. Formally, the rules are as follows

Stage 1: Simultaneously the players choose to commit to actions in A_1 and A_2, respectively, or to wait till stage 2.

Stage 2: Each player i who did not yet choose an action in A_i is informed about what action his opponent j took in stage 1. After having received this information, the player is required to choose an action in A_i with players moving simultaneously if both still have to make a choice.

Payoffs: The players' actions in γ^2 lead to a unique outcome in A. If $a \in A$ results, then player i's payoff in γ^2 is $u_i(a)$. Hence, there is no discounting, moving late does not entail any cost. Also, there is no cost involved in moving early.

Hence, the question to be addressed in the remaining sections is whether only commitment robust equilibria of g correspond to "sensible" equilibrium outcomes of γ^2. We game each equilibrium in undominated strategies is perfect (Van Damme (1987, Theorem 3.2.2.)). Since $\bar{a}_i$ is undominated in g and g satisfies (6.3.1), $\bar{a}_i$ can be dominated in g^2 only if it is dominated by a strategy in which player i moves only in period 2. Suppose this strategy tells player i to play the mixed action s_i in the second period in case the other player also waited. Let $a_j, a'_j \in A_j$ and consider a strategy for player j that tells him to wait till period 2 and then to play a_j if player i committed to $\bar{a}_i$, and to play a'_j otherwise. It follows that $u_i(\bar{a}_i, a_j) \leq u_i(s_i, a'_j)$ for all $a_j, a'_j \in A_j$. Since each player has a unique best reply against each pure strategy, substitution of $a_j = a'_j = \bar{a}_j$ implies that $s_i = \bar{a}_i$. Hence, $u_i(\bar{a}_i, a_j) = u_i(\bar{a})$ for all $a_j \in A_j$. This contradicts (6.3.1). □

At this stage it is appropriate to compare our work with Hamilton and Slutsky (1990). Hamilton and Slutsky consider the game g^2 where g is the standard quantity setting Cournot duopoly game. In their Theorem VIII they claim

> "the two Stackelberg equilibria are the only pure strategy equilibria in undominated strategies. Playing the Cournot equilibrium strategy at the first turn is dominated by waiting to play after one's rival."

This claim obviously is inconsistent with Theorem 6.4. Now it is certainly true that the standard Cournot duopoly game does not fit our context and discretized versions of this game need not satisfy (6.3.1). Nevertheless, it is not difficult to see that the above claim is wrong. Playing the Cournot equilibrium strategy at the first stage would indeed be dominated if players could be sure that they would continue with the Cournot equilibrium in the second stage in case both players still have to move, and if each player could be sure that the opponent would always best respond to a unilateral commitment. However, players cannot be sure of this since the logic of the perfectness concept forces them to consider the possibility of mistakes in the second period. The total quantity in the second period might be above the Cournot quantity, if only by mistake, and in this case it pays to commit to the Cournot quantity if the opponent does the same. It is our impression that, although not stating it explicitly, Hamilton and Slutsky actually had this truncated game (that excludes second period mistakes) in mind when making the above claim. (See also their companion paper Hamilton and Slutsky (1993) in which in the proof of Theorem III they display the payoff matrix of the truncated game rather than of the full game g^2.)

Although the relevance of the truncated game may be questioned — after all, what is the rationale for allowing mistakes in the first period of γ^2 but excluding them in the second period? — we can state a result about it that generalizes the observation of

Hamilton and Slutsky about the quantity setting duopoly game: An equilibrium s of the original game is an outcome of a perfect equilibrium of this truncated game if and only if it is a CRE. If s is pure, the proof of this result is straightforward. In the case s is in mixed strategies, the proof follows the same lines as that of Theorem 6.2. Formally, if $s^* \in S$, write $\gamma^2(s^*)$ for the extensive form game that results from γ^2 if we force each player to best respond in period 2 to a unilateral commitment of the opponent in period 1 and if we replace the subgame w at stage 2 in which both players still have to move by an endpoint with payoff vector $(u_1(s^*), u_2(s^*))$. Let $g^2(s^*)$ be the associated normal form. We have

Theorem 6.5 *s^* is a commitment robust equilibrium if and only if there exist a perfect equilibrium σ of $g^2(s^*)$ with outcome s^*.*

Proof. See Proposition 4 in Van Damme and Hurkens (1993). □

6.6 Endogenous timing and coordination

In this section we investigate whether more refined equilibrium notions allow one to draw the conclusion that only commitment robust equilibria are viable when the order of the moves is endogenous. Specifically, we address the question of whether in a game that has a unique and pure CRE, endogenous timing will force the players to coordinate on this CRE. We will show that some set-valued "evolutionary" concepts, viz. the notions of persistent equilibria (Kalai and Samet (1984)) and of curb* equilibria (Basu and Weibull (1991)) do indeed allow this conclusion. For the definitions and properties of these concepts the reader is referred to Chapter 2.

Before we come to the main result of this chapter, we state

Lemma 6.2 *In g^2 no player has equivalent strategies among his pure undominated ones.*

Proof. Suppose on the contrary that player i has equivalent strategies among his pure undominated ones in g^2. Hence, there exist $a_i, \bar{a}_i \in A_i$ such that either (i) a_i^1 is equivalent with $\bar{a}_i^1$, or (ii) a_i^1 is equivalent with $\bar{a}_i^2$ or (iii) a_i^2 is equivalent with $\bar{a}_i^2$. In all cases it follows that $u_i(a_i, a_j) = u_i(\bar{a}_i, a_j')$, for all $a_j, a_j' \in A_j$. This implies that $u_i(a_i, a_j) = u_i(a_i, a_j')$, for all $a_j, a_j' \in A_j$. But this contradicts (6.3.1). □

The main result of this chapter is[3]

[3]Since any RCRE is a CRE, this theorem remains valid when "CRE" is replaced by "RCRE".

Theorem 6.6 *Assume $\bar{a}$ is a pure CRE of g. Then*

(i) Any curb, resp. persistent equilibrium of g^2 yields each player i a payoff of at least $u_i(\bar{a})$.*

(ii) If $\bar{a}$ is the unique CRE of g, then each curb, resp. persistent equilibrium of g^2 results in the outcome $\bar{a}$.*

Proof. In this proof, let "x" stand for "curb*" or "persistent". By Lemma 6.2 and Theorem 2.1(ii) we know that x-retracts are convex hulls of pure undominated strategies and different x-retracts are disjoint. Let the retract $\bar{R}$ be defined by

$$\bar{A}_i = \{\bar{a}_i^1\} \cup \{a_i^2 : \; a_i \in A_i\}, \;\; \bar{R}_i = \Delta \bar{A}_i, \;\; \bar{R} = \bar{R}_1 \times \bar{R}_2.$$

Note that $\bar{R}$ is closed under undominated best replies and absorbing. We will show that any x-retract is contained in $\bar{R}$.

The proof is easy in case some player i has a dominant strategy $\tilde{a}_i$ in g. Then $\tilde{a}_i = \bar{a}_i$ and we have that for all $a_i \neq \bar{a}_i$, a_i^2 is dominated by $\bar{a}_i^2$. Moreover, for all $a_j \in A_j$ and all $a_i \neq \bar{a}_i$ we have $U_i(\bar{a}_i^1, a_j^1) > U_i(a_i^1, a_j^1)$ (since $\bar{a}_i$ is dominant in g) and $U_i(\bar{a}_i^1, a_j^2) > U_i(a_i^1, a_j^2)$ (since $\bar{a}_i$ is player i's unique Stackelberg leader strategy). This implies that if R is an x-retract, then

$$R \subset \Delta(\{\bar{a}_i^1, \bar{a}_i^2\}) \times \Delta(\bar{A}_j) \subset \bar{R}.$$

Now assume that no player has a dominant strategy in g. Then it is easily seen that $\bar{a}_i^2$ is an undominated strategy for each player i. Furthermore, for each j there exists some $\tilde{a}_j$ such that $\bar{a}_i \notin B(\tilde{a}_j)$. We will show that, if $a_j \neq \bar{a}_j$, then committing to a_j cannot belong to any x-retract. Note that if R is an x-retract and $a_j^1 \in R_j$, then $\bar{a}_i^2 \in R_i$. Namely, $\bar{a}_i^2$ is an undominated best response against a_j^1 (hence $\bar{a}_i^2 \in R_i$ if x stands for curb*, and $\bar{a}_i^2$ is the unique best response against $(1-2\varepsilon)a_j^1 + \varepsilon \tilde{a}_j^1 + \varepsilon \tilde{a}_j^2$ (so that $\bar{a}_i^2 \in R_i$ if x stands for persistent). Now, since $\bar{a}_j^2$ is the unique best response against $(1-\varepsilon)\bar{a}_i^2 + \varepsilon \tilde{a}_i^1$ it follows that $(\bar{a}_1^2, \bar{a}_2^2) \in R$ whenever $a_i^1 \in R_i$ for some $a_i \neq \bar{a}_i$. Since $\bar{R}$ is closed under undominated best replies and absorbing, we have that any x-retract that contains $(\bar{a}_1^2, \bar{a}_2^2)$ is contained in $\bar{R}$ and there does not exist an x-retract containing some a_i^1 with $a_i \neq \bar{a}_i$. Hence, any x-retract is contained in $\bar{R}$.

Now note that, if players are restricted to choose strategies from $\bar{R}$, then each player i can guarantee the payoff $u_i(\bar{a})$ by playing $\bar{a}_i^1$. Consequently, if σ is an x-equilibrium of g^2, then $C(\sigma) \subset \bar{R}$ and $u_i(\sigma) \geq u_i(\bar{a})$ for $i = 1, 2$, which proves the first part of the theorem.

Now assume that there exists an x-equilibrium σ that results in an outcome different from $\bar{a}$. Then for each player i we must have $\sigma_i(\bar{a}_i^1) < 1$. Define the mixed strategy s_i of player i in g by

$$s_i(a_i) = (1 - \sigma_i(\bar{a}_i^1))^{-1}\sigma_i(a_i^2) \qquad (a_i \in A_i)$$

Since σ is an equilibrium of g^2 we have that s is an equilibrium of g and, furthermore $u_i(s) \geq \bar{u}_i$ for $i = 1, 2$, since each player i can guarantee $\bar{u}_i$ by playing $\bar{a}_i^1$. Hence, s is a CRE of g. This completes the proof. □

6.7 Conclusion and related literature

We have addressed the question of whether only equilibria at which no player has an incentive to move first are viable when the order of the moves is endogenously determined and players have the opportunity to commit themselves. We have seen that in order to answer this question in the affirmative one needs quite strong equilibrium concepts. However, if one is willing to accept such concepts, one can indeed conclude that endogenous timing forces players to coordinate on a unique and pure commitment robust equilibrium. We have restricted ourselves in this chapter to 2-person games and we have only allowed one point in time at which a player can commit himself. It is important to investigate the extent to which our results depend on these assumptions. One can easily define the game γ^t in which there are $t-1$ periods in which a player has the opportunity to commit. (Obviously $\gamma^1 = g$.) The reader can verify that our main results remain valid in this extended context. Hence, mixed equilibria are typically not viable and any curb* (resp. persistent) equilibrium of the game in which the players have $t-1$ opportunities to commit themselves results in the commitment robust equilibrium. We have not investigated whether our results extend to games with more than two players, although we expect they do. It is clear, however, that in some cases (as in the proof of Theorem 6.2) different techniques are needed.

For convenience we assumed throughout the chapter regularity condition (6.3.1) to hold. It is worthwhile to remark that Theorems 6.3, 6.4 and 6.5 also hold under the milder regularity assumptions

$$u_i(a) = u_i(a') \text{ if and only if } u_j(a) = u_j(a') \quad (i \neq j) \tag{6.7.1}$$

and

$$|B(a_i)| = 1 \text{ for all } a_i \in A_i \quad (i = 1, 2) \tag{6.7.2}$$

However, for Theorem 6.6 to hold we then need the additional assumption that players have a unique Stackelberg leader strategy. To see why the latter assumption (which is also implied by (6.3.1)) is essential, consider game g in Figure 6.3. Note that this game satisfies (6.7.1) and (6.7.2). (T, L) is a CRE in this game, however, the unique curb* (or persistent) retract of g^2 contains all first period commitment strategies, so that, in particular (B^1, R^1) is a curb* equilibrium.

	L		C		R	
T	3	3	0	0	0	0
M	2	2	3	3	0	0
B	0	0	4	1	2	2

Figure 6.3.

We conclude with discussing some related papers. We refrain from discussing the (rather extensive) literature that is built on the idea that, if the equilibrium associated with a certain timing of the moves Pareto dominates all equilibria associated with any alternative sequencing, then the Pareto optimal sequencing will result. (See the literature that is cited in Hamilton and Slutsky (1990) in relation with "the extended game with observable delay".) We have already discussed Rosenthal (1991) and Hamilton and Slutsky (1990) in the Sections 6.3 and 6.5. In Hamilton and Slutsky (1993), the action commitment game is studied for the special case where g is a 2×2 game in which exactly one player (say player 1) has a dominant strategy. Obviously, g has a unique equilibrium s^* in this case. Hamilton and Slutsky consider the reduced game $g^2(s^*)$ introduced in Section 6.5 and they show that this has a unique perfect equilibrium. By the above Theorem 6.6 the outcome of this equilibrium is s^* when s^* is a CRE. If s^* is not a CRE, then the outcome is that player 1 commits to his dominated strategy. d'Aspremont and Gérard-Varet (1980) define a Nash equilibrium s to be a Stackelberg equilibrium if $u_i(s) = u_i^+$ for $i = 1, 2$. As we have already seen, a Stackelberg equilibrium is an RCRE, but the converse need not hold. They point out that in a strictly competitive game, each Nash equilibrium is a Stackelberg equilibrium. (Cf. Lemma 6.1, where we actually proved this slightly stronger result.) They also show that each "twisted equilibrium" that is Pareto optimal is a Stackelberg equilibrium. (s is a twisted equilibrium if $u_i(s) \leq u_i(s_i, s_j')$ for all s_j' and all $i \neq j$, hence, if player i does not lose when j deviates.) d'Aspremont and Gérard-Varet do not discuss endogenous timing of the moves. Such aspects are discussed in two papers by Robson. Robson (1989) considers a 2-stage game that is closely related to the action commitment game that has been analyzed in this chapter. The two differences are that mixed strategies can be communicated and that moving early entails a cost. Specifically, if the outcome s results and i moves in period t ($t \in \{1, 2\}$), then player i's payoff is equal to $u_i = u_i(s) - (2-t)c_i$ where $c_i > 0$. Robson's most important result is that, if each Nash equilibrium is a Stackelberg equilibrium (as defined in d'Aspremont and Gérard-Varet (1980)), then all pure subgame perfect equilibria of the timing game involve choosing a Stackelberg equilibrium. (The result follows easily from the observation that there cannot be a pure equilibrium in which both players move early.) It is worthwhile to point out that this result depends essentially on the restriction to pure strategies. Namely, consider the game from Figure 6.1a and assume that moving in period 1 costs $\varepsilon > 0$. Now (B^1, R^1) is no longer an equilibrium, but there is a mixed equilibrium close to it, namely each player choosing $(0, \varepsilon, 2\varepsilon, 1-3\varepsilon)$. Hence, in this equilibrium each player commits immediately to "1,1" with probability $1 - 3\varepsilon$; if the game reaches the second stage without any commitments being made, players play the mixed equilibrium of the underlying game. Robson (1990) generalizes the model from Robson (1989) by allowing for more periods in which the players can commit themselves. It is assumed that the earlier the player moves, the costlier it is and it is shown that under certain conditions the timing game has a unique pure subgame perfect equilibrium. The equilibrium outcome is that the player who gains most from leading indeed becomes the leader. Again the result depends crucially on the restriction to pure strategies. Another drawback of Robson's model is that the cost functions are specified exogenously, i.e. it is not explained why there would be a cost of moving early. For the special class of common interest games, Balkenborg (1993) has shown that Theorem 6.6 continues to hold for another set-valued evolutionary concept, viz. that of direct evolutionary stable sets, abbreviated DESS. Balkenborg shows that a set E of mixed strategy vectors is a DESS if and only if

$$E = \cup_{\sigma \in E}(\mathcal{B}(\sigma_2) \times \{\sigma_2\}) = \cup_{\sigma \in E}(\mathcal{B}(\sigma_1) \times \{\sigma_1\})$$

and he proves that a DESS of g^2 exists if and only if g is a common interest game. Furthermore, in case g has common interest, there is a unique DESS and it consists of all strategy combinations yielding the Pareto efficient payoff vector.

In the game g^2 it is assumed that, if a player commits himself, this commitment is perfectly observed by the opponent. Bagwell (1992) considers the case where only one player can commit himself but where this commitment is only imperfectly observed and he claims that in this case the strategic benefits of the commitment possibility evaporate. As we will show in Chapter 8, Bagwell's result depends essentially on the restriction to pure strategies: When mixed strategies are taken into account, there is continuity in the sense that all "sensible" equilibria of the noisy game converge to the Stackelberg equilibrium of the game without noise. We believe that a similar continuity result can be established in the present setting with two-sided commitment possibilities.

It has to be admitted that the class of games with a commitment robust equilibrium, i.e. the class for which we were able to determine the outcome with endogenous timing in this chapter is quite limited. In the next chapter we address the question of which outcomes can be expected for some economic games without a commitment robust equilibrium.

Appendix

This Appendix provides the proofs of Theorems 6.1 and 6.2.

Proof of Theorem 6.1. Consider the game γ^{ij} in which player i acts as a Stackelberg leader, with this player's mixed action being revealed to player j. Because of the bilinearity of u_i, player i's payoff if he chooses s_i and j best responds, is at most $\max_{a_j \in B(s_i)} u_i(s_i, a_j)$. Hence, each SPE yields player i at most u_i^+. If $u_i(s^*) = u_i^+$, then any strategy pair σ with $\sigma_i = s_i^*$ and σ_j with $\sigma_j(s_i^*) = s_j^*$ and $\sigma_j(s_i) \in B(s_i)$ for all s_i is an SPE of γ^{ij}. This proves the first part of the theorem.
To prove the second part, let $s_i^+ \in S_i$ and $a_j^+ \in B(s_i^+)$ be such that $u_i(s_i^+, a_j^+) = u_i^+$. Because of (6.3.4), there exists $x \in \mathbf{R}^{A_i}$ with $x(a_i) = 0$ if $a_i \notin C(s_i^+)$ such that $u_j(x, a_j^+) > 0$ and $u_j(x, a_j) = 0$ for all $a_j \in B(s_i^+)\backslash\{a_j^+\}$. (Here $u_j(x, a_j)$ is shorthand notation for $\sum_{a_i} x(a_i) u_j(a_i, a_j)$.) Write $t = \sum_{a_i} x(a_i)$ and $s_i(\varepsilon, t) = (1 + \varepsilon t)^{-1}(s_i^+ + \varepsilon x)$. Then $s_i(\varepsilon, t) \in S_i$ if ε is sufficiently small, $B(s_i(\varepsilon, t)) = \{a_i^+\}$ and $s_i(\varepsilon, t) \to s_i^+$ as $\varepsilon \to 0$. Hence, if s is an SPE outcome of γ^{ij} we must have $u_i(s) \geq u_i(s_i(\varepsilon, t), a_j^+)$, and, therefore, $u_i(s) \geq u_i(s_i^+, a_j^+) = u_i^+$. This completes the proof of the second part of the theorem. □

Before we give the proof of Theorem 6.2, it will be convenient to first state a lemma that lists some properties of games satisfying (6.3.4).

Lemma 6.3 *Let g satisfy (6.3.4) and let s be an equilibrium of g that is not pure. Then*

(i) $|C(s_1)| = |C(s_2)| = |B(s_1)| = |B(s_2)|$

(ii) If s' is an equilibrium and $C(s') = C(s)$, then $s' = s$.

(iii) For each $a_i \in C(s_i)$, there exists $a_j \in C(s_j)$ such that $a_j \notin B(a_i)$.

(iv) For each $a_i \in C(s_i)$, there exists $a_j \in C(s_j)$ such that $a_i \notin B(a_j)$.

Proof.

(i) Since s is an equilibrium, we have $C(s_i) \subset B(s_j)$ for $i, j \in \{1,2\}$. Since the rank of a matrix cannot exceed the number of columns, (6.3.4) implies $|B(s_j)| \leq |C(s_j)|$ for all s_j. Combining these observations yields $|C(s_1)| \leq |B(s_2)| \leq |C(s_2)| \leq |B(s_1)| \leq |C(s_1)|$, hence, all inequalities must be equalities.

(ii) Assume s and s' are equilibria with $C(s) = C(s')$. If $u_i(s) = 0$, then the columns of the matrix u_{is_j} are dependent, hence, (6.3.4) is violated. So assume $u_i(s) \neq 0$. Consider the vector v with a_j–th coordinate equal to $v(a_j) = u_i(s')s_j(a_j) - u_i(s)s'_j(a_j)$. Then, if we premultiply v by u_{is_j} we get zero, hence by (6.3.4) v must be the zero vector. Therefore, $u_i(s')s_j = u_i(s)s'_j$ and since both s_j and s'_j are probability vectors $u_i(s') = u_i(s)$. But then $s_j = s'_j$. A similar argument implies that $s_i = s'_i$.

(iii) This follows immediately from the fact that $|B(a_i)| = 1$ and $|C(s_j)| \geq 2$.

(iv) If $a_i \in B(a_j)$ for all $a_j \in C(s_j)$, then $\{a_i\} = B(a_j)$ for all such a_j, hence $\{a_i\} = B(s_j)$ contradicting (i). □

Before we come to the proof of Theorem 6.2 we need to introduce some notation. Assumption (6.3.1) forces player j, in any SPE, to play $b(a_i)$ in the second period when player i committed himself unilaterally to a_i. Hence, an SPE strategy σ_i of player i only needs to tell which action from $A_i \cup \{w_i\}$ to choose in stage 1 and what to do in stage 2 in the information set $w = (w_1, w_2)$ that corresponds to the case where both players waited. For such a strategy, we write $\sigma_i = (\sigma_i^1, \sigma_i^2)$ where σ_i^1 denotes the randomization at time 1 and σ_i^2 is the mixed action in information set w. For the case where $\sigma_i^1(w_i) < 1$, i.e. player i moves with positive probability in the first period, it will also be convenient to write

$$\omega_i = \sigma_i^1(w_i),\ s_i^1 = (1-\omega_i)^{-1}\sigma_i^1,\ \text{and}\ s_i^2 = \sigma_i^2 \tag{A.1}$$

hence, s_i^1 is the mixed action that player i plays if he moves in period 1. In this case we will also write $\sigma_i = (\omega_i, s_i^1, s_i^2)$. Writing

$$\delta_i(a) = \begin{cases} 1 & \text{if}\ \ a_i = b(a_j) \\ 0 & \text{otherwise,} \end{cases} \tag{A.2}$$

we see that the outcome p^σ of an SPE σ is given by

$$p^\sigma(a) = \sigma^1(a) + (1-\omega_1)\omega_2\sigma_1^1(a_1)\delta_2(a) + \omega_1(1-\omega_2)\sigma_2^1(a_2)\delta_1(a) + \omega_1\omega_2\sigma^2(a), \tag{A.3}$$

where we have used $\sigma^t(a)$ as a shorthand notation for $\sigma_1^t(a_1)\sigma_2^t(a_2)$.

Proof of Theorem 6.2.

Sufficiency. No player has an incentive to move first at s^* if and only if $u_i(s^*) \geq \bar{u}_i$ for $i = 1, 2$. In this case the strategy pair $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_i^1 = w_i$, $\sigma_i^2 = s_i^*$ is a subgame perfect equilibrium of γ^2 with outcome s^*.

Necessity. Assume $u_i(s^*) < \bar{u}_i$ for some i and that σ is a subgame perfect equilibrium of γ^2 with outcome s^*. We will derive a contradiction. Without loss of generality we may assume $u_1(s^*) < \bar{u}_1$. The proof is divided into a number of steps.

Step 1. $\omega_i = \sigma_i^1(w_i) < 1$ for $i = 1, 2$, hence, each player moves with positive probability in the first period.

Clearly, we must have $\sigma_2^1(w_2) < 1$, for otherwise player 1 can guarantee the payoff $\bar{u}_1$ by committing in the first period, hence $U_1(\sigma) \geq \bar{u}_1$ and σ cannot have the outcome s^*. Suppose that player 1 waits till the second period, i.e. $\sigma_1^1(w_1) = 1$. Let a^* be such that $\sigma_2^1(a_2^*) > 0$ and $a_1^* \in C(s_1^*)\backslash B(a_2^*)$. Then the outcome a^* can only be obtained if it is played in the second period, hence, $p^\sigma(a^*) \leq \omega_2 s^*(a^*)$, a contradiction.

Step 2. $\omega_i = \sigma_i^1(w_i) > 0$ for $i = 1, 2$, hence, with positive probability each player waits till the second period of γ^2 in σ.

Namely, assume $\sigma_i^1(w_i) = 0$ for some i. Then $\sigma_i^1(a_i) = s_i^*(a_i)$ for all a_i and i chooses at least two strategies with positive probability (Lemma 6.3). Furthermore, since there does not exist $a_j \in A_j$ with $a_j \in B(a_i)$ for all $a_i \in C(s_i^*)$, we have $U_j(\sigma_i, a_j^2) > U_j(\sigma_i, a_j^1)$ for all a_j, hence, $\sigma_j^1(w_j) = 1$. This contradicts step 1.

From the steps 1 and 2 we can conclude that the stage-2 subgame (w_1, w_2) is reached with positive probability if σ is played in γ^2. Since σ is a subgame perfect equilibrium of γ^2, σ^2 must be a Nash equilibrium of g. Furthermore, obviously $C(\sigma^2) \subset C(s^*)$. We will distinguish two cases:

Case A $C(\sigma^2) = C(s^*)$, hence (by Lemma 6.3) $\sigma^2 = s^*$.

Case B $C(\sigma^2) \neq C(s^*)$, hence (by Lemma 6.3), there exists for each player i a strategy a_i^* such that $a_i^* \in C(s_i^*) \backslash C(\sigma_i^2)$.

We first continue the proof for case A.

Step A3. For each player i: If $\sigma_i^1(a_i) > 0$, then $u_i(a_i) > u_i(s^*)$, hence, a player commits himself only to strategies that yield more than the equilibrium payoff.

Suppose, to the contrary, that there exists some $\bar{a}_i$ such that $\sigma_i^1(\bar{a}_i) > 0$ but $u_i(\bar{a}_i) \leq u_i(s^*)$. Since player i must be indifferent between $\bar{a}_i^1$ and a_i^2 for any $a_i \in C(s^*)$, we must have

$$\sum_{a_j} \sigma_j^1(a_j) u_i(\bar{a}_i, a_j) + \omega_j u_i(\bar{a}_i) = \sum_{a_j} \sigma_j^1(a_j) u_i(a_j) + \omega_j u_i(s^*)$$

hence

$$\begin{aligned} u_i(\bar{a}_i) &= u_i(s^*) \quad \text{and} \\ u_i(\bar{a}_i, a_j) &= u_i(a_j) \quad \text{for all } a_j \in C(\sigma_j^1). \end{aligned}$$

In particular, $\bar{a}_i$ must be a best response against any pure action that player j plays with positive probability in the first period. From Lemma 6.3 we can conclude that there exists $a_j^* \in C(s_j^*)$ such that $\bar{a}_i \notin B(a_j^*)$ and $\sigma_j^1(a_j^*) = 0$. We may assume that $a_j^* \in B(\bar{a}_i)$, since otherwise by (A.3)

$$p^\sigma(\bar{a}_i, a_j^*) = \omega_1 \omega_2 s^*(\bar{a}_i, a_j^*) < s^*(\bar{a}_i, a_j^*).$$

Applying Lemma 6.3 once more we see that we can find a_i^* such that $a_j^* \notin B(a_i^*)$. Then, for $a^* = (a_1^*, a_2^*)$,

$$p^\sigma(a^*) = \omega_1 \omega_2 s^*(a^*) < s^*(a^*),$$

another contradiction.

Step A4. Conclusion of the proof for case A.

We have seen up to now that each player must move in both periods with positive probability and that for each player i and for each action a_i with $\sigma_i^1(a_i) > 0$ we have $u_i(a_i) > u_i(s^*)$. Since $U_i(\sigma) = u_i(s^*)$ and since waiting is a best response for each player we have

$$\sum_{a_j} \sigma_j^1(a_j) u_i(a_j) + \omega_j u_i(s^*) = u_i(s^*)$$

so that there must exist a_j^* with $\sigma_j^1(a_j^*) > 0$ and $u_i(a_j^*) \leq u_i(s^*)$. Hence, $\sigma_j^1(a_j^*) > 0$ and $u_i(a_i, a_j^*) \leq u_i(s^*)$ for all $a_i \in A_i$, and, therefore

$$\sigma_j^1(a_j^*) > 0 \quad \text{and} \quad a_j^* \notin B(a_i) \quad \text{for all } a_i \text{ with } \sigma_i^1(a_i) > 0.$$

Consider $a^* = (a_1^*, a_2^*)$. Then $p^\sigma(a^*) = \sigma^1(a^*) + \omega_1\omega_2 s^*(a^*)$, hence, since $p^\sigma = s^*$, we must have

$$\sigma^1(a^*) = (1 - \omega_1\omega_2) s^*(a^*). \tag{A.4}$$

On the other hand, for each player i, we have by summing up (A.3) over all a_j, $p^\sigma(a_i^*) = \sigma_i^1(a_i^*) + \omega_1\omega_2 s_i^*(a_i^*)$, hence

$$\sigma^1(a^*) = (1 - \omega_1\omega_2)^2 s^*(a^*). \tag{A.5}$$

Combining (A.4) with (A.5) and using $a^* \in C(s^*)$ yields $\omega_1\omega_2 \in \{0, 1\}$. But this contradicts the steps 1 and 2 and completes the proof for Case A.

We now continue with the proof of case B. Let a^* be such that, for each player i, $a_i^* \in C(s_i^*) \backslash C(\sigma_i^2)$.

Step B3. For all players i, j and all actions, if $a_j \in C(s_j^*)$ and $a_j \notin B(a_i^*)$, then $a_i^* \in B(a_j)$.

Assume there exists $\bar{a}_j \in C(s_j^*)$ such that $\bar{a}_j \notin B(a_i^*)$ and $a_i^* \notin B(\bar{a}_j)$. In σ the outcome $(a_i^*, \bar{a}_j)$ can occur only if both players play it in the first period, hence

$$s^*(a_i^*, \bar{a}_j) = \sigma^1(a_i^*, \bar{a}_j). \tag{A.6}$$

By summing up (A.3) over all a_l it follows that, for $k \neq l$, $s_k^*(a_k) = p^\sigma(a_k) \geq \sigma_k^1(a_k)$. Combining this with (A.6) and (A.3) it follows that

$$\sigma_i^1(a_i^*) = s_i^*(a_i^*) \qquad \text{and} \qquad a_i^* \notin B(a_j) \text{ if } \sigma_j^1(a_j) > 0.$$

This implies that any outcome (a_i^*, a_j) with $a_j \notin B(a_i^*)$ can occur only in the first period, hence,

$$s^*(a_i^*, a_j) = \sigma^1(a_i^*, a_j) \qquad \text{for all } a_j \notin B(a_i^*),$$

and hence, $s_j^*(a_j) = \sigma^1(a_j)$ for $a_j \notin B(a_i^*)$. This implies that if player j acts in period two, then he plays the unique $\bar{a}_j \in B(a_i^*)$ for sure. But this implies (by Lemma 6.3(i)) that $\bar{a}_j \in B(a_i)$ for all $a_i \in C(s_i^*)$. This contradicts Lemma 6.3(iv).

Step B4. Conclusion of the proof for case B.

Consider the pair $a^* = (a_1^*, a_2^*)$. Without loss of generality, step B3 allows us to assume that $a_1^* \in B(a_2^*)$. By Lemma 6.3 we have

$$\text{if } a_1 \in C(s_1^*),\ a_1 \neq a_1^*, \text{ then } a_1 \notin B(a_2^*).$$

By step B3, therefore $a_2^* \in B(a_1)$ for all such a_1. But then we have that $a_2^* \in B(\sigma_1^2)$ and $a_2^* \notin C(\sigma_2^2)$. By Lemma 6.3 this is impossible since σ^2 is an equilibrium of g and g satisfies the regularity condition (6.3.4). □

Chapter 7

Endogenous Timing and Risk Dominance

7.1 Introduction

In the previous chapter, we investigated endogenous timing in finite bimatrix games. We showed that some solution concepts with an evolutionary flavor (in particular, persistent and curb* equilibria) select the commitment robust equilibrium, whenever it is unique and pure. However, we did not provide much insight in games that do not have commitment robust equilibria. As is well known, both players have an incentive to move first in the standard Cournot duopoly model. Hence, this game has no commitment robust equilibrium and the previous chapter does not tell us what will happen when timing is determined endogenously.

Several authors have addressed the issue of timing in economic games before. Many of these papers investigate first and/or second mover advantages. (See Ono (1982), Gal-Or (1985), Dowrick (1986) and Boyer and Moreaux (1987)). This literature examines for which parameters there exists a particular ordering of moves that is preferred by all players.

In this chapter we will again employ the two period action commitment game introduced by Hamilton and Slutsky (1990). The reason is that we want the order of moves be determined endogenously. We will investigate some economic games with a unique Nash equilibrium. The action commitment game then has three subgame perfect equilibria in pure strategies: Either both players play the Nash equilibrium in period one or the players play in different periods. Hamilton and Slutsky claimed that only the two Stackelberg equilibria are in undominated strategies. As we showed in the previous chapter, however, playing the Nash equilibrium in period one is also an undominated strategy. We will make a unique selection among the pure subgame perfect equilibria on the basis of risk considerations. Namely, committing in period one is quite risky, since the other player may commit himself at the same time, and there is the possibility of Stackelberg warfare. We will characterize the risk dominant equilibrium (in the sense of Harsanyi and Selten (1988)) in three specific games: (1) the Cournot quantity competition, (2) the (Bertrand) price competition with differentiated (substitutable) products, and (3) the private provision of a public good.

Harsanyi and Selten (1988) defined the risk dominance relation between two equilibria s and s' in terms of the *bicentric prior* and the *tracing procedure*. The bicentric prior is a vector of initial beliefs of the players when there is uncertainty whether s or s' is the solution of the game. (We will see later how the bicentric prior has to be computed.) The tracing procedure models the reasoning process of players who try to figure out what to play. At the start of the tracing procedure each player computes the best reply against his prior. Then he gradually puts more and more weight on the actual strategy of the opponent, and adjusts his best reply accordingly. Eventually, he will put all weight on the actual strategy of the opponent and play optimally against this belief. Hence, players end up playing an equilibrium. Now equilibrium s is said to risk dominate s' if the tracing procedure, starting from the bicentric prior based on s and s', ends up in s.

The formal definitions of the bicentric prior and the tracing procedure are quite complex, as we will see later. However, Harsanyi and Selten gave a simple characterization of the risk dominant equilibrium in 2×2 games. They showed that the equilibrium, at which the so called product of the deviation losses is largest, is risk dominant. As an example, consider the game in Figure 7.1 and compare the pure equilibria (T, L) and (B, R). The product of the deviation losses at (T, L) is equal to $(9-8)(9-8) = 1$. The product of the deviation losses at (B, R) is $(7-0)(7-0) = 49$. Hence, (B, R) risk dominates (T, L) and (B, R) is called the risk dominant equilibrium.

	L	R
T	9,9	0,8
B	8,0	7,7

Figure 7.1.

For larger games, such a simple characterization of the risk dominant equilibrium has not been obtained. Most applications of risk dominance have therefore been restricted to 2 × 2 games.[1] In this chapter we will employ the original notion of risk dominance that is based on the bicentric prior and the tracing procedure. We will also compare the risk dominant equilibrium with the equilibrium at which the product of the deviation losses is largest. We will see that they do not always coincide.

It is obvious that there is no risk dominance relation between the two Stackelberg equilibria when the players are completely symmetric. We therefore assume that players (or firms) differ in their (constant marginal) costs. We obtain the following results. First, we show that both Stackelberg equilibria risk dominate the simultaneous move equilibrium. Second, we show that the Stackelberg equilibrium with the low cost firm as the leader, risk dominates the other Stackelberg equilibrium. This result is intuitive in the case of Cournot competition. The low cost firm obtains the preferred role. This result was already conjectured by Von Stackelberg (1934, p.66), who said that the most modern firm should be the leader. However, in the case of price competition with differentiated substitutable products, our result seems to be counterintuitive: In this situation each firm prefers to be a follower to being a leader. Hence, it may happen, when the costs are not too different, that the high cost firm makes a higher profit (as a follower) than the low cost firm (as a leader). Also in the case of the private provision of a public good, our result seems to be counterintuitive, because it selects the inefficient equilibrium. However, we select on the basis of risk considerations and not on the basis of efficiency. The message of this chapter is that committing is risky, but it is less risky for low cost firms.

The rest of this chapter is organized as follows. In Section 7.2 we explain the notion of risk dominance. We adjust the definition from Harsanyi and Selten (1988) in order to deal with games in which the strategy sets are not finite. We also briefly recall the rules of the action commitment game. Section 7.3 contains the results. Section 7.4 concludes. Tedious proofs are to be found in the Appendix.

7.2 Preliminaries

In this chapter we work with games with infinite strategy sets. Notation will therefore differ from that introduced in Section 1.3. Throughout this chapter, $g = (A_1, A_2, u_1, u_2)$ is a two person game where the pure strategy set is $A_i = [0, \infty)$, and where $u_i : A_j \times A_i \to$

[1] For a brief discussion of the exceptions see Van Damme (1994b).

$\mathbf{R}$ is "sufficiently" differentiable. We assume that for each $a_j \in A_j$ the function

$$a_i \mapsto u_i(a_j; a_i)$$

is concave and has a unique maximizer, denoted $B_i(a_j)$. We also assume that the function

$$C(a_j) = u_j(B_i(a_j); a_j)$$

has a unique maximizer, denoted a_j^L, which is obtained by the first order condition $C'(a_j^L) = 0$. We refer to a_j^L as player j's Stackelberg leader strategy, and to $a_j^F = B_j(a_i^L)$ as his Stackelberg follower strategy. Furthermore, we assume that g has a unique (interior) Nash equilibrium, a^N, which is in pure strategies. Let

$$N_i = u_i(a_j^N; a_i^N), \quad L_i = u_i(a_j^F; a_i^L), \quad F_i = u_i(a_j^L; a_i^F)$$

denote i's payoff in the Nash equilibrium, as a Stackelberg leader and as a Stackelberg follower, respectively. Note that $L_i \geq N_i$. We assume that at least one player has an incentive to move first at a^N, i.e. $L_i > N_i$ for some $i \in \{1,2\}$. Hence, a^N is not commitment robust.

As in the previous chapter, we let γ^2 denote the two period action commitment game. We briefly recall the rules of this game. There are two periods and each player has to move in exactly one of these periods. Choices are made simultaneously, but, if one player chooses to move early, (i.e. to commit) while the other moves late, the latter is informed about the former's choice before making his decision. Our assumptions imply that the proper subgames of γ^2 are easily solved. Namely, the subgame that is reached when both players wait has a unique equilibrium, a^N. Furthermore, in the subgames that are reached when there is a unilateral commitment in period 1, there is a unique best reply for the player who waited.

In this chapter we will apply elements from the equilibrium selection theory of Harsanyi and Selten (1988) to the action commitment game. Their theory prescribes that first the solutions of the smallest subgames must be determined. As we stated before, this is very easy in our action commitment game, since all proper subgames of γ^2 already have a unique solution. Let $g^2(a^N)$ denote the game that results from γ^2 by substituting these solutions. According to the theory of Harsanyi and Selten we now have to determine the solution of $g^2(a^N)$. This is then also the solution of γ^2.

We may write $g^2(a^N) = (A_1^*, A_2^*, u_1^*, u_2^*)$, where

$$A_i^* = A_i \cup \{W_i\},$$

and where, for all $a \in A$

$$\begin{aligned} u_i^*(a_j; a_i) &= u_i(a_j; a_i) \\ u_i^*(W_j; a_i) &= u_i(B_j(a_i); a_i) \\ u_i^*(a_j; W_i) &= u_i(a_j; B_i(a_j)) \\ u_i^*(W_j; W_i) &= u_i(a_j^N; a_i^N) \end{aligned}$$

Hence, in $g^2(a^N)$ the strategy a_i means "player i commits himself to a_i in period 1", while W_i means "player i waits till period two and best responds to j's commitment, if there is one, and plays a_i^N if j did not commit".

Two remarks are in order here concerning the game $g^2(a^N)$. First, it is easy to see that $g^2(a^N)$ has three pure equilibria: Either both players play in period one (in which case they must play a^N), or players move in different periods. The pure equilibria are therefore

$$N := (a_1^N, a_2^N), \quad S_1 := (a_1^L, W_2), \text{ and } \quad S_2 := (W_1, a_2^L).$$

Second, note that against each strategy $a_j^* \in A_j^*$, player i is not worse off by waiting than by playing a_i^N. If a_i^N is not dominant in g (as we will assume from now on), then a_i^N is weakly dominated by W_i. In fact, each strategy $a_i \in A_i$ for which $u_i(B_j(a_i); a_i) \leq N_i$, is weakly dominated by W_i.

According to the theory of Harsanyi and Selten, we now have to determine the set of candidate solutions of $g^2(a^N)$. The candidate solutions are the primitive (or curb) equilibria of $g^2(a^N)$. Of course, in Chapter 2 we defined the concept of curb sets only for finite games, but it can be easily extended to games with infinite strategy spaces. (See Basu and Weibull (1991).) It turns out that curb sets do not have much bite here, but curb* sets do. Namely, if a_i^F is weakly dominated by W_i, then $\{S_j\}$ is closed under undominated best replies (but not under best replies). Since, W_i is an undominated best reply against each strategy $a_j \in A_j$, we can have three different situations:

(i) If a_i^F is weakly dominated by W_i for $i = 1, 2$, then $g^2(a^N)$ has two minimal curb* sets, namely $\{S_1\}$ and $\{S_2\}$.

(ii) If a_i^F is dominated by W_i, but a_j^F is not dominated by W_j, then $g^2(a^N)$ has a unique minimal curb* set, namely $\{S_j\}$.

(iii) If a_i^F is not dominated by W_i, then $g^2(a^N)$ has a unique minimal curb* set, including S_1 and S_2, and possibly some other, mixed, equilibria. Since a_i^N is weakly dominated, we know that N cannot be an element of this minimal curb* set.

According to this slightly adjusted version of Harsanyi and Selten's theory, in situation (ii) the solution is S_j. In situation (i) a selection has to be made between the Stackelberg equilibria, while in situation (iii) a selection has to be made between the Stackelberg equilibria and the mixed equilibria in the minimal curb* set. Note that in all cases the equilibrium N is excluded from the set of candidates.

In this chapter we will diverge from the theory of Harsanyi and Selten. Namely, we will take the set of candidates to be $\{N, S_1, S_2\}$. The main reason for this is that the final selection between the candidates will be based on the notion of risk dominance, which is in turn based on the tracing procedure. The tracing procedure is a process that, starting from some given prior beliefs of the players, gradually adjusts players' plans and expectations until they are in equilibrium. It models the thought process of players, who, by deductive personal reflection, try to figure out what to play. On the other hand, primitive solutions (here, the curb* equilibria) are relevant in an evolutionary context, where the game is repeatedly played by a large population of players who receive feedback from the evolution of play during the game. (See Chapter 3.[2]) Hence, it seems that the theory of Harsanyi and Selten mixes arguments from an evolutionary context with arguments from a deductive context.

We have some additional reasons for taking $\{N, S_1, S_2\}$ as the candidate set. Namely, it is quite easy to show that N is risk dominated by S_1 and S_2 (Theorem 7.1). Therefore, we can exclude N as the solution of the game solely on the basis of risk considerations. Then we only need to compare the Stackelberg equilibria. This comparison has to be made also in cases (i) and (iii) if only primitive solutions are candidates. The reason for excluding mixed strategy equilibria from the candidate set is simply that they are very difficult to compute in games with infinite strategy sets.

As mentioned before, the final selection between the candidate solutions will be based on the notion of the risk dominance relation.[3] Risk dominance is defined in terms of the *bicentric prior* and the *tracing procedure*. Before we can formally speak about priors, we need to introduce some notation for beliefs over the (infinite) strategy spaces A_i^* $(i = 1, 2)$.

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[2]Of course, Chapter 3 only deals with finite games.

[3]According to Harsanyi and Selten (1988), payoff dominance should precede risk dominance. In our action commitment game it may happen that S_1 and S_2 payoff dominate N. (This is the case in a differentiated product market where prices are the strategic variables.) Harsanyi and Selten would then eliminate N on the basis of payoff dominance. However, we are more interested in the risk associated with committing, and will therefore select on the basis of risk dominance only. In Section 7.3 it is shown that N is always risk dominated by S_1 and S_2.

To that end, we need a topology on A_i^*. We extend the Euclidean topology on A_i to A_i^* in a natural way: A set $T \subset A_i^*$ is called open if and only if $T \cap A_i$ is open in the Euclidean topology on A_i. Now we call any probability measure μ_j on A_i^* a belief for player j. Such a belief can be decomposed into a pair (β_j, F_j), where $\beta_j = \mu_j(W_i^*)$ is the probability that j assigns to i playing W_i, and where $F_j/(1-\beta_j)$ is a conditional distribution function over A_i (conditional on i not playing W_i).

Now we can describe the mechanics of the tracing procedure. Let $\mu = (\mu_1, \mu_2)$ be a pair of (prior) beliefs. For any $t \in [0,1]$ we define the strategic form game $g^{t,\mu} = (A_1^*, A_2^*, u_1^{t,\mu}, u_2^{t,\mu})$ as follows:

$$\begin{aligned} u_1^{t,\mu}(a_2^*; a_1^*) &= t u_1^*(a_2^*; a_1^*) + (1-t)\int_{A_2^*} u_1^*(a_2; a_1^*) d\mu_1 \\ u_2^{t,\mu}(a_1^*; a_2^*) &= t u_2^*(a_1^*; a_2^*) + (1-t)\int_{A_1^*} u_2^*(a_1; a_2^*) d\mu_2 \end{aligned}$$

Hence, for $t = 1$ this game coincides with $g^2(a^N)$, while for $t = 0$ we have a trivial game in which each player's payoff depends only on his prior and his own action, but not on the action of his opponent. Write $\Gamma^{t,\mu}$ for the graph of the equilibrium correspondence, i.e.

$$\Gamma^{t,\mu} = \{(t; s) | t \in [0,1], \quad s \text{ is an equilibrium of } g^{t,\mu}\}.$$

In the case of nondegenerate *finite* games it can be shown that this graph contains a unique distinguished curve that connects the unique equilibrium of $g^{0,\mu}$ to an equilibrium of $g^{1,\mu}$. (See Schanuel et al. (1991) for details.) For infinite games as the ones we are interested in here, however, such results have not been obtained. In the examples we will consider in this chapter, there is always a distinguished curve. The equilibrium of $g^{1,\mu}$ on this curve is the outcome of the tracing procedure.

Now we define the bicentric prior $\mu^0(s, s')$, based on two equilibria s and s' of $g^2(a^N)$. Harsanyi and Selten (1988) have in mind the situation in which it is common knowledge among the players that either s or s' is the solution of the game. Each player i will initially assume that his opponent j already knows which of the two is the solution. Player i will assign a subjective probability z_i to the solution being s (and, hence, to j playing s_j) and he will assign the complementary probability $1 - z_i$ to j playing s'_j. After having constructed these beliefs, i will play a best response $b_i(z_i)$ against $z_i s_j + (1-z_i)s'_j$. (For convenience, assume that this best reply $b_i(z_i)$ is unique for almost all z_i.) Player j does not know i's belief z_i, and, according to the principle of insufficient reason, j will assume that z_i is uniformly distributed on $[0,1]$. Hence, j's belief about player i will be of the form

$$\mu_j^0(s, s') = (\beta_j, F_j), \tag{7.2.1}$$

where

$$\beta_j = \lambda(\{z \in [0,1] | b_i(z) = W_i\})$$

and where

$$\begin{aligned} F_j(a_i) &= Prob(i \text{ plays some strategy in } [0, a_i]) \\ &= \lambda(\{z \in [0,1] | b_i(z) \in [0, a_i]\}), \end{aligned}$$

where λ denotes Lebesgue measure.

Note that player j's expected payoff from playing $a_j^* \in A_j^*$, when his belief is given by the prior $\mu_j^0(s, s')$, is equal to

$$\begin{aligned} \int_{A_i^*} u_j(a_i; a_j^*) d\mu_j^0 &= \beta_j u_j^*(W_i; a_j^*) + \int_{A_i} u_j^*(a_i; a_j^*) dF_j \\ &= \beta_j u_j^*(W_i; a_j^*) + \int_{b_i^{-1}(A_i)} u_j^*(b_i(z); a_j^*) dz \end{aligned}$$

For this reason it will be more convenient to write the prior $\mu_j^0(s, s')$ as

$$\beta_j W_i + \int_{b_i^{-1}(A_i)} b_i(z) dz,$$

although this is with some abuse of notation. Note that, if player j has this prior, where $\beta_j = 0$ and where $b_i(z)$ is not a constant function, then his best reply is to wait. Namely, in that case j is sure that i will commit himself, but he is not sure to which strategy.

Risk dominance is now defined as follows. Equilibrium s risk dominates another equilibrium s' if the outcome of the tracing procedure, starting with the bicentric prior μ^0 as defined in (7.2.1), is s. In case the outcome of the tracing procedure is an equilibrium different from s and s', then neither of the equilibria risk dominates the other.

7.3 Economic games of timing

In this section we consider the risk dominance relation between the three pure equilibria of $g^2(a^N)$. Our first result states that playing the Nash equilibrium in stage one is risk dominated by both Stackelberg equilibria. In order to determine which one of the Stackelberg equilibria risk dominates the other, we restrict attention to three special examples.

In particular, we will investigate a standard Cournot duopoly model, a (Bertrand) price competition model with differentiated products, and a model of private contributions to a public good. When players are completely symmetric, there is no reason why one Stackelberg equilibrium should be preferred over the other. Therefore, we will introduce an asymmetry between the players: We assume that firms differ with respect to marginal costs.

Theorem 7.1

If player i has an incentive to move first at a^N, then S_i risk dominates N.

Proof. Without loss of generality we just prove that S_1 risk dominates N when player 1 has an incentive to move first. Note that $L_1 > N_1$ implies that $a_1^N \neq a_1^L$ and $a_2^N \neq a_2^F$. First we have to calculate the bicentric prior.

Suppose player 1 plays $za_1^L + (1-z)a_1^N$. Since $a_2^N \neq a_2^F$, we have that for all $z \in (0,1)$ player 2's best reply is to wait till period two. Hence, player 1 believes that player 2 will wait till period two with probability one, and his best reply is to commit to a_1^L.

Now suppose player 2 plays $zW_2 + (1-z)a_2^N$. For every z player 1 can guarantee his Nash equilibrium payoff by waiting but also by committing to his Nash strategy. However, player 1's best reply is to commit to some $a_1(z) \neq a_1^N$, for all $z > 0$: For the optimal commitment strategy $a_1(z)$ it holds that

$$zC'(a_1(z)) + (1-z)\left(\frac{\partial u_1}{\partial a_1}\right)|_{(a_2^N;a_1(z))} = 0, \tag{7.3.1}$$

where $C(a_1) = u_1(B_2(a_1); a_1)$. In particular, $a_1(0) = a_1^N$ and $a_1(1) = a_1^L$. Suppose that there is a $\bar{z} > 0$ such that $a_1(\bar{z}) = a_1^N$. Then it must hold that $a_1(z) = a_1^N$ for all $z \in [0, \bar{z}]$. Differentiation of (7.3.1) with respect to z at $z = 0$ yields $C'(a_1^N) = 0$. But this is impossible since $a_1^N \neq a_1^L$. Hence, player 1 will commit with probability one, but player 2 is not sure to which strategy. Hence, the best reply of player 2 is to wait. Hence, the tracing procedure already starts at $t = 0$ with the Stackelberg equilibrium S_1. But then the players stick to this equilibrium from $t = 0$ to $t = 1$. □

The above result is quite intuitive. In the (reduced) game $g^2(a^N)$, a_i^N is weakly dominated by waiting. It is of course very risky to play such a strategy.

Now we will analyze which one of the Stackelberg equilibria risk dominates the other. We will have to restrict attention to some very specific examples. Even in these simple examples it is already quite complicated to compute the outcome of the tracing procedure.

7.3.1 Cournot duopoly

There are two firms, 1 and 2. Firm i produces a quantity $q_i \geq 0$ at a constant marginal cost $c_i > 0$. The market price is linear: $p = A - q$, where $q = q_1 + q_2$. The profit of firm i is then $u_i(q_j; q_i) = (A - q_1 - q_2 - c_i)q_i$. We assume that $3c_i - 2c_j \leq A$ (all $i, j \in \{1, 2\}$). This assumption implies that a Stackelberg follower will not be driven out of the market. Let $\alpha_i = A - c_i$. We assume that $c_1 > c_2$.

Now it is easy to check that the best reply (or reaction function) of player j is $B_j(q_i) = \max\{(\alpha_j - q_i)/2, 0\}$. Equilibrium actions and payoffs are given by

$$a_i^N = \frac{2\alpha_i - \alpha_j}{3}, \quad a_i^L = \frac{2\alpha_i - \alpha_j}{2}, \quad a_i^F = \frac{3\alpha_i - 2\alpha_j}{4}$$

$$N_i = \frac{(2\alpha_i - \alpha_j)^2}{9}, \quad L_i = \frac{(2\alpha_i - \alpha_j)^2}{8}, \quad F_i = \frac{(3\alpha_i - 2\alpha_j)^2}{16}$$

In Figure 7.2a the reaction functions are drawn and the equilibrium quantities are indicated. It is easily checked that

$$L_i > N_i > F_i, \qquad (i = 1, 2)$$

so that both players have an incentive to move first. Since a_i^F is weakly dominated by W_i, $g^2(a^N)$ has two minimal curb* sets, namely $\{S_1\}$ and $\{S_2\}$. According to Harsanyi and Selten only the Stackelberg equilibria are candidate solutions. As we argued before, we also consider N to be a candidate.

From Theorem 7.1 we already know that S_1 and S_2 risk dominate N. The solution of the game is thus found by comparing S_1 and S_2, and we obtain the same solution as Harsanyi and Selten. Von Stackelberg conjectured that the most modern firm (the one with the lower costs) will emerge as the leader, hence, he conjectured that S_2 will be played. We will confirm his conjecture by showing that S_2 risk dominates S_1.

We first have to compute the bicentric prior based on S_1 and S_2. Suppose player i plays $za_i^L + (1 - z)W_i$. The optimal commitment strategy of player j is then to commit to

$$q_j(z) = \frac{z(3\alpha_j - 2\alpha_i) + (1 - z)(2\alpha_j - \alpha_i)}{2(z + 1)} = \frac{\alpha_j}{2(z + 1)} + \frac{\alpha_j - \alpha_i}{2}.$$

Note that $q_j(0) = a_j^L$ and $q_j(1) = a_j^F$ and that $c_1 > c_2$ implies that $q_2(z) > q_1(z)$ for all z. Of course, committing to this strategy is only optimal if it yields a higher profit than waiting. Committing to $q_j(z)$ yields

$$\pi_j^c(z) = \frac{1}{8(z + 1)}(z(3\alpha_j - 2\alpha_i) + (1 - z)(2\alpha_j - \alpha_i))^2,$$

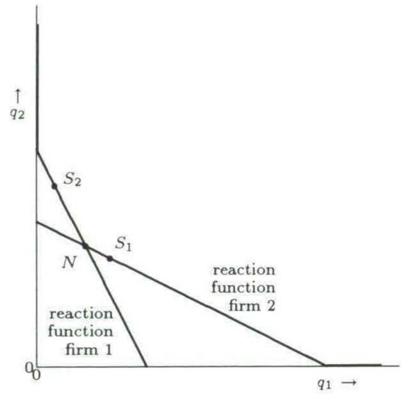

Figure 7.2a: Reaction functions.

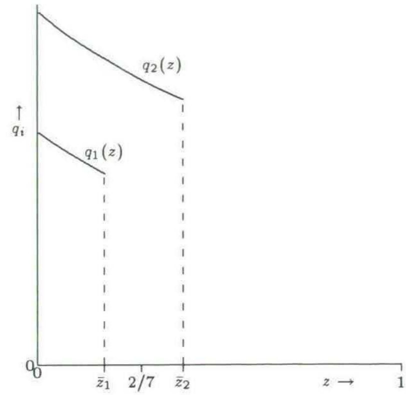

Figure 7.2b: Bicentric prior.

while waiting yields

$$\pi_j^w(z) = \frac{z}{16}(3\alpha_j - 2\alpha_i)^2 + \frac{1-z}{9}(2\alpha_j - \alpha_i)^2.$$

It is optimal for player j to commit only if z is not too high. More specifically, committing is optimal for player j as long as $z \in [0, \bar{z}_j)$, where

$$\bar{z}_j = \frac{(4\alpha_j - 2\alpha_i)^2}{18\alpha_j^2 - (4\alpha_j - 2\alpha_i)^2}.$$

This implies that the bicentric prior of player i is given by

$$\mu_i^0(S_1, S_2) = (1 - \bar{z}_j, F_j) = (1 - \bar{z}_j)W_j + \int_0^{\bar{z}_j} q_j(z)dz.$$

The priors are indicated in Figure 7.2b. From these priors we can develop the intuition for why it is the low cost firm that will become the leader in the risk dominant equilibrium. Namely, first note that our assumptions on the cost parameters imply that $2/3\alpha_2 \leq \alpha_1 < \alpha_2$, and, hence, that $1/17 < \bar{z}_1 < 2/7 < \bar{z}_2 \leq 32/49$. This tells us that firm 1 assigns a higher probability to the event that firm 2 will commit than that firm 2 assigns to the event that firm 1 will commit. Furthermore, note that $q_2'(z) < q_1'(z) < 0$. This implies that the conditional distribution $F_2/\bar{z}_2$ has a larger variance than $F_1/\bar{z}_1$. This tells us that firm 1 is more uncertain about the commitment strategy of the opponent than firm 2. Both observations suggest that committing is more risky for the high cost firm.

In principle, we can have 4 different situations concerning the best replies of the players against their prior: (i) Both players prefer to wait, (ii) Only player 1 prefers to commit,

(iii) Only player 2 prefers to commit and (iv) Both players prefer to commit. However, below we will show that player 2 always (that is, for all allowed parameter values) prefers to commit, while player 1 only prefers to commit in case the difference in cost is not too large (Lemma 7.1).[4] This implies that situations (i) and (ii) never occur. In situation (iii) it is obvious that the tracing procedure selects S_2, since player 1 commits and player 2 waits from $t = 0$ till $t = 1$ in the tracing procedure. We will show that in situation (iv) player 1 will be the first to switch to his waiting strategy (Lemma 7.2 and 7.3). Hence, also in this situation S_2 is selected.

We define the following function

$$\begin{aligned}
G_j(t, b, q_i; q_j) \;&:=\; (1-t)\int_0^b \{u_j(q_i(z); q_j) - u_j(q_i(z); B_j(q_i(z)))\}dz \\
&\quad +(1-t)(1-b)(u_j(B_i(q_j); q_j) - u_j^N) \qquad\qquad (7.3.2)\\
&\quad +t(u_j(q_i; q_j) - u_j(q_i; B_j(q_i)) \\
&= \;(1-t)\int_0^b q_j(\alpha_j - q_j - q_i(z)) - \frac{1}{4}(\alpha_j - q_i(z))^2 dz \\
&\quad +(1-t)(1-b)(q_j\frac{2\alpha_j - \alpha_i - q_j}{2} - (\frac{2\alpha_j - \alpha_i}{3})^2) \\
&\quad +t(q_j(\alpha_j - q_j - q_i) - \frac{1}{4}(\alpha_j - q_i)^2)
\end{aligned}$$

This function denotes the (expected) gain of player j from committing to q_j instead of waiting, in the tracing procedure at time t, when player i actually commits to q_i and player j's prior is $(1-b)W_i + \int_0^b q_i(z)dz$. In particular, $G_j(t, \bar{z}_i, q_i; q_j)$ is player j's gain from committing in the tracing procedure at time t, when we start from the bicentric prior.

For all $t \in [0, 1]$, let $(\tilde{q}_1(t), \tilde{q}_2(t))$ be the pair of optimal commitment strategies in the tracing procedure starting from the bicentric prior at time t, i.e.[5]

$$\{\tilde{q}_j(t)\} = \arg\max_q G_j(t, \bar{z}_i, \tilde{q}_i(t); q). \qquad (j = 1, 2)$$

We write $\tilde{G}_j(t) = G_j(t, \bar{z}_i, \tilde{q}_i(t); \tilde{q}_j(t))$ for the expected gain from committing optimally. In particular, player j prefers to commit at t if $\tilde{G}_j(t) > 0$ and prefers to wait if $\tilde{G}_j(t) < 0$.

[4]Some intuition for this result can be obtained by computing the "worst case" scenario's for player 2, where $c_1 = c_2$ and $\bar{z}_1 = 2/7$, and for player 1, where $\alpha_1 = 2\alpha_2/3$ and $\bar{z}_2 = 32/49$. In the worst case of player 2 (which is equivalent to the best case for player 1), committing yields a higher payoff than waiting. In the worst case for player 1, it is optimal to wait. The remaining problem is to show that committing becomes more attractive when the opponent's cost increases.

[5]The Appendix provides exact expressions for $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$.

Lemma 7.1 *(i)* $\tilde{G}_2(0) > 0$.

(ii) $\tilde{G}_1(0) > 0$ *if* $\alpha_2/\alpha_1 < 1.0805$ *and* $\tilde{G}_1(0) < 0$ *if* $\alpha_2/\alpha_1 > 1.081$.

Proof. See Appendix.

Hence, firm 2 always prefers to commit at the start of the tracing procedure. Firm 1 prefers to commit only for some parameter values. We assume from now on that the parameters are such that $\tilde{G}_1(0) \geq 0$. Our next lemma states that, as long as player 1 (weakly) prefers to commit, player 2 (strictly) prefers to commit.

Lemma 7.2 *Let* $t^* = \max\{t' \in [0,1] | \tilde{G}_1(t) \geq 0 \textit{ for all } t \in [0,t']\}$. *If* $t^* > 0$ *then* $\tilde{G}_2(t) > \tilde{G}_1(t)$ *for all* $t \in [0, t^*)$.

The formal proof is relegated to the Appendix. We give a sketch of the proof here. It is based on comparisons between payoffs in the tracing procedure starting with the bicentric prior, and payoffs obtained in the tracing procedure that starts with the alternative prior $(1-\bar{z}_2)W_i + \int_0^{\bar{z}_2} q_i(z)dz$ for each player j. Hence, in this alternative tracing procedure player 1 starts with the same (bicentric) prior as before, but player 2 starts with a more pessimistic prior: he overestimates the probability with which player 1 will commit. Let $(\hat{q}_1(t), \hat{q}_2(t))$ be the optimal commitment strategies in the alternative tracing procedure, i.e.

$$\{\hat{q}_j(t)\} = \arg\max_q G_j(t, \bar{z}_2, \hat{q}_i(t); q). \qquad (j = 1,2)$$

In the Appendix we prove that $\tilde{q}_2(t) > \hat{q}_2(t)$ and $\tilde{q}_1(t) < \hat{q}_1(t)$. These inequalities are intuitive: When player 2 is more pessimistic, he will be more cautious and commit to a lower quantity. Consequently, player 1 will commit to a higher quantity in that case.

The lemma follows now from six inequalities:

$$\begin{aligned} G_2(t, \bar{z}_1, \tilde{q}_1(t); \tilde{q}_2(t)) &\geq G_2(t, \bar{z}_1, \tilde{q}_1(t); \hat{q}_2(t)) \geq G_2(t, \bar{z}_2, \tilde{q}_1(t); \hat{q}_2(t)) \\ &\geq G_2(t, \bar{z}_2, \hat{q}_1(t); \hat{q}_2(t)) > G_1(t, \bar{z}_2, \hat{q}_2(t); \hat{q}_1(t)) \\ &\geq G_1(t, \bar{z}_2, \hat{q}_2(t); \tilde{q}_1(t)) \geq G_1(t, \bar{z}_2, \tilde{q}_2(t); \tilde{q}_1(t)) \end{aligned}$$

The first inequality is by definition of $\tilde{q}_2(t)$. The second inequality follows from the pessimism of player 2. The third inequality follows since $\hat{q}_1(t) > \tilde{q}_1(t)$. The intuition for the fourth inequality is that both players are equally pessimistic about the probability with which the other will commit. But since $q_2'(z) < q_1'(z) < 0$, we have that $q_2(0) -$

$q_2(\bar{z}_2) > q_1(0) - q_1(\bar{z}_2)$. Hence, player 1 is more uncertain about which strategy the other player will commit himself. The fifth inequality follows from the definition of $\tilde{q}_1(t)$. The last inequality follows from the fact that $\tilde{q}_2(t) > \hat{q}_2(t)$.

The above two lemma's are almost sufficient to show that player 1 will be the first to give in and switch to his waiting strategy. We just need that for the t^* from Lemma 7.2 it holds that $t^* < 1$. That is, we just need to show that it is impossible that both players will keep committing during the whole tracing procedure. If they would, the tracing procedure would result in playing N. Fortunately, we have

Lemma 7.3 *There exists some $t \in (0,1)$ such that $\tilde{G}_1(t) < 0$.*

Proof. It is easy to see that $\tilde{q}_j(1) = a_j^N$ $(j = 1,2)$ and that $\tilde{G}_1(1) = 0$. The lemma follows now from

$$\begin{aligned}
\tilde{G}_1'(1) &= -\int_0^{\bar{z}_2} \{u_1(q_2(z); a_1^N) - u_1(q_2(z); B_1(q_2(z)))\} dz - 0 \\
&\quad +0 + \tilde{q}_1'(1)(\alpha_1 - 2a_1^N - a_2^N) + \tilde{q}_2'(1)(\frac{1}{2}\alpha_1 - a_1^N - \frac{1}{2}a_2^N) \\
&= -\int_0^{\bar{z}_2} \{u_1(q_2(z); a_1^N) - u_1(q_2(z); B_1(q_2(z)))\} dz > 0
\end{aligned}$$

□

This completes the proof of

Theorem 7.2 *In the Cournot duopoly model, the Stackelberg equilibrium with the low cost firm as the leader and the high cost firm as the follower risk dominates the Stackelberg equilibrium where the roles are reversed.*

To conclude this subsection we compare the products of deviation losses for both Stackelberg equilibria. In order to do that, we need to know the payoff for the players in case of Stackelberg warfare, that is, when both players choose their Stackelberg leader strategy. It is easily computed that i's Stackelberg warfare payoff equals

$$D_i = \frac{1}{4}(\alpha_i - \alpha_j)(2\alpha_i - \alpha_j).$$

The product of deviation losses at S_i is given by $(L_i - N_i)(F_j - D_j)$. Substitution of the previously computed payoffs yields that the product of deviation losses at S_2 is larger than the one at S_1.

7.3.2 Price setting duopoly with differentiated products

Assume there are two firms producing differentiated (substitutable) products. Firm i produces good i at constant marginal cost c_i. We assume that firms compete in prices, and that demand for good i is given by

$$q_i = 1 - p_i + ap_j,$$

where $0 < a < 1$. The profit of firm i, when prices p_1 and p_2 are chosen, is therefore $u_i(p_j; p_i) = (p_i - c_i)(1 - p_i + ap_j)$. We will assume that $1 > c_1 > c_2 > 0$. It is easy to verify that $B_j(p_i) = (1+ap_i+c_j)/2$. From this one easily verifies that $F_i > L_i > N_i$, that is, both firms prefer to be followers, but of course they prefer to be a leader to playing simultaneously. Reaction functions and equilibrium prices are indicated in Figure 7.3a.

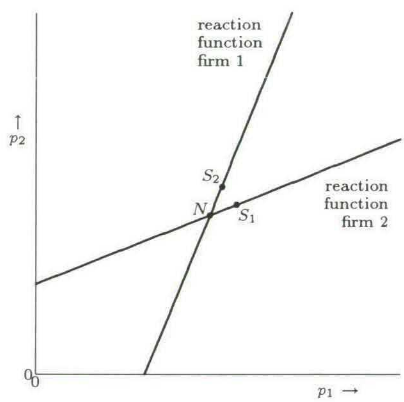

Figure 7.3a: Reaction functions.

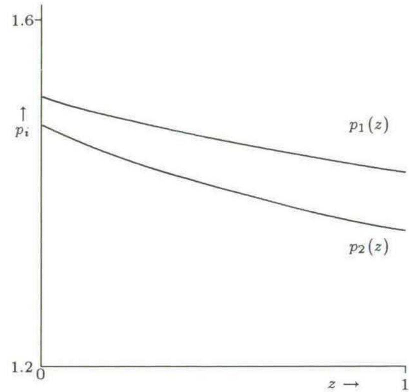

Figure 7.3b: Bicentric prior.

For further reference, we give the simultaneous Nash equilibrium payoffs

$$N_i = \frac{(2 + a + ac_j + (a^2 - 2)c_i)^2}{(4 - a^2)^2}.$$

The reader might suspect that the high cost firm will become the leader, so that the low cost firm obtains the preferred role. However, it will turn out that the low cost firm will be the leader in the risk dominant equilibrium. Note that, if the costs differ only slightly, the firm with the high cost makes a higher profit as a follower than the firm with the low cost does as a leader. The intuition for this result is that it is risky to commit yourself. The low cost firm faces a smaller risk than the high cost firm, and therefore he will take the leader role, although he would prefer the follower role. The formal proof consists of a number of straightforward, but very tedious steps.

First we have to consider the bicentric prior based on S_1 and S_2. Suppose that player i plays $zp_i^L + (1-z)W_i$. The best player j can do is to commit to a certain price $p_j(z)$, where

$$p_j(z) = \frac{4 + 2a - 2a^2 - a^3 + 2ac_i - a^3c_i + 4c_j - 4a^2c_j + a^4c_j + a^2z(1 + a + (3-a^2)c_j)}{2(2-a^2)(a^2z + 2 - a^2)}.$$

Obviously, $p_j(0) = p_j^L$ and $p_j(1) = p_j^F$. Note that committing to p_j^F is already strictly better than waiting, for all $z < 1$. Hence, the bicentric prior for player i is given by

$$\mu_i^0(S_1, S_2) = \int_0^1 p_j(z)dz.$$

Both players believe that the other player will commit with probability one, but they are not sure to what price the other player will commit himself. Therefore, they prefer to wait and see. Hence, the tracing procedure starts at $t = 0$ with both players waiting. As time passes, players put more and more weight on the event that the other player waits, and eventually one player will give in and move first. We will show that the low cost firm will give in first. From the bicentric prior we can develop the intuition for this result: It is easy to check that $p_2'(z) < p_1'(z) < 0$. (See also Figure 7.3b.) Hence, firm 1 is more uncertain about the opponent's commitment strategy, than firm 2 is. Hence, committing is more risky for the high cost firm.

Let $C_i(t,p)$ denote player i's payoff at time t if he commits to p while the other player still waits and best responds in period two, i.e.

$$C_i(t,p) = (1-t)\int_0^1 u_i(p_j(z);p)dz + tu_i(B_j(p);p).$$

Let $\tilde{p}_i(t)$ denote the optimal commitment strategy, i.e. $\tilde{p}_i(t) = \arg\max C_i(t,\cdot)$ for all t. It is not difficult to verify that

$$\tilde{p}_i(t) = \frac{(1-t)(1 + c_i + a\int_0^1 p_j(z)dz) + t(2 + a + ac_j + (2-a^2)c_i)/2}{2 - a^2t}.$$

Let $W_i(t)$ denote player i's payoff from waiting at time t, i.e.

$$W_i(t) = (1-t)\int_0^1 u_i(p_j(z); B_i(p_j(z)))dz + tN_i.$$

Let $G_i(t) = C_i(t, \tilde{p}_i(t)) - W_i(t)$ denote player i's gain from committing. Note that $G_i(0) < 0 < G_i(1)$. As long as player j has not switched yet, player i will switch at time $t = \hat{t}_i$, where $G_i(\hat{t}_i) = 0$. Substitution of $\tilde{p}_i(\hat{t}_i)$ in the latter equation and multiplying both sides by $2 - a^2\hat{t}_i$ yields $P_i(\hat{t}_i) = 0$, where

$$\begin{aligned} P_i(t) \quad = \quad & \frac{1}{8}(2 - 2c_i + 2a\int_0^1 p_j(z)dz + t(a + ac_j + a^2c_i - 2a\int_0^1 p_j(z)dz))^2 \\ & -(2-a^2t)((1-t)\int_0^1 u_i(p_j(z); B_i(p_j(z)))dz + tN_i) \end{aligned}$$

$P_i(t)$ is a polynomial of degree 2 (or less) and we already know that it has a root between 0 and 1. We need to show that $\hat{t}_2 < \hat{t}_1$. Since the polynomial is of degree 2 (or less) an explicit solution could be obtained. The coefficients of this polynomial are very messy and depend on the parameters a, c_1 and c_2, and this direct method will therefore not render a solution to our problem. Therefore, we consider the difference polynomial $D(t) = P_1(t) - P_2(t)$. In the Appendix we prove

Lemma 7.4 *$D(t) < 0$ for all $t \in [0,1]$ and all $a \in (0,1)$.*

From this it then follows that $\hat{t}_2 < \hat{t}_1$. Consequently, we proved

Theorem 7.3 *In the price setting duopoly with differentiated products, the Stackelberg equilibrium with the low cost firm as a leader and the high cost firm as the follower risk dominates the Stackelberg equilibrium in which the roles are reversed.*

This result is surprising and counterintuitive. Namely, it is well-known that the follower makes a higher profit than the leader when both firms have the same cost. If the cost differential is positive, but not too large, this remains true. In this case, the low cost firm would have an incentive to *increase* his cost. Theorem 7.3 can also be viewed as an illustration of the fact that it may be beneficial to be "weak", that is, to have high cost.

We conclude this subsection by computing the product of the deviation losses at each Stackelberg equilibrium. It is easily verified that

$$\begin{aligned} L_i - N_i &= \frac{a^4(2 + a + ac_j + (a^2-2)c_i)^2}{8(2-a^2)(4-a^2)^2} \\ F_j - D_j &= \frac{a^4(1 + ac_i - c_j)^2}{16(a^2-2)^2} \end{aligned}$$

It follows that the product of the deviation losses at S_1 is larger than at S_2 if and only if

$$(c_1 - c_2)(2 + 2a + (c_1 + c_2)(a^2 - 1)) > 0.$$

Hence, the product of deviation losses is larger at S_1 than at S_2, although S_2 is the risk dominant equilibrium.

7.3.3 Private provision of a public good.

We have two players, 1 and 2. Each player i can contribute to a public good, at a constant marginal cost c_i. When contributions a_1 and a_2 are chosen, player i's payoff is

$$u_i(a_1, a_2) = \sqrt{a_1 + a_2} - c_i a_i.$$

We assume that $2c_2 > c_1 > c_2 > 0$.

Let $d_j = (2c_j)^{-2}$. It is easily verified that the best reply function for player j is given by $B_j(a_i) = \max\{d_j - a_i, 0\}$. It easily follows that $a_1^N = 0$, $a_2^N = d_2$, $a_j^L = 0$ and $a_j^F = d_j$. Reaction functions and equilibrium contributions are indicated in Figure 7.4.

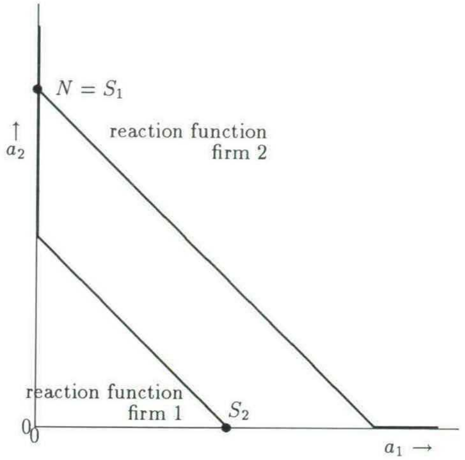

Figure 7.4: Reaction functions.

We see that the leader wants to commit to contribute nothing. This game is a bit peculiar since the Nash equilibrium outcome coincides with the Stackelberg outcome when player 1 is the leader. Theorem 7.1 can not be applied to exclude equilibrium N, since player 1 has no incentive to move first. The timing game has a continuum of (mixed) equilibria in which player 1 commits to contribute nothing, and player 2 mixes between contributing d_2 in period 1 and waiting. Denote the equilibrium in which player 2 plays in period 1 with probability λ by $s^*(\lambda)$. Hence, $s^*(\lambda) = \lambda N + (1-\lambda)S_1$. Note that it is very easy to compare the products of the deviation losses of S_2 and $s^*(\lambda)$. Namely, the deviation loss for player 2 at $s^*(\lambda)$ is zero, because he is indifferent. Since the product of the deviation losses at S_2 is strictly positive, S_2 has the largest product.

Theorem 7.4 *S_2 risk dominates $s^*(\lambda)$ for all $\lambda \in [0,1]$.*

Proof. Suppose player 2 plays $zs_2^*(\lambda) + (1-z)a_2^L$. For all $z \in (0,1)$ player 1 has a unique best reply, namely to wait. Hence, player 2 believes that player 1 will wait with probability 1, and his best reply is to commit to contribute nothing.

Now suppose that player 1 plays $zs_1^*(\lambda) + (1-z)W_1 = za_1^L + (1-z)W_1$. The unique best reply for player 2 is to commit to contribute $z^2 d_2$, for all $z < 1$. Hence, player 1 is almost sure that player 2 will commit, but he is not sure to which action. Consequently, player 1's best reply to his prior is to wait.

Now it is clear that the tracing procedure will lead to S_2: On the whole path from $t = 0$ to $t = 1$ player 2 commits in period 1 and player 1 waits till period 2. □

Remark. It seems that S_1 is more credible. If player 1 commits to contribute nothing, and player 2 responds by contributing d_2, then ex post player 1 still does not want to contribute. However, if player 2 commits to contribute nothing, and player 1 responds by contributing d_1, then the situation is different. Ex post player 2 would like to contribute $d_2 - d_1$ (and player 1 would not mind!). The assumption that players can credibly commit is very strong here. It must be absolutely impossible to renege on your commitment, even if everybody would profit from such change of action.

7.4 Conclusion

In this chapter we endogenized the timing of the moves in some interesting economic games without a commitment robust equilibrium. We made a selection between the three subgame perfect equilibria on the basis of risk dominance. To our knowledge, this is the first attempt to apply the tracing procedure in games where the strategy spaces are not finite.[6] The reader might wonder whether the selection on basis of risk considerations has any predictive power. After all, the tracing procedure is quite complicated. It seems very unlikely that players actually will compute the outcome of this procedure. However, the tracing procedure is meant to model the thought process of players. Players need not be aware of how their thought process works. In any case, we believe that the bicentric prior gives some idea about the risks that different players face, and that the risk dominant equilibrium is more likely to prevail.

In this chapter we restricted ourselves to pure strategy equilibria (except for the case of a public good). We did this merely for convenience. However, we suspect that equilibria in which players play in both periods with positive probability will be risk dominated. We also restricted ourselves to three specific games. Of course, there are many other interesting economic games without a commitment robust equilibrium. For instance, one could consider the quantity and price model of this chapter where the products are complements instead of substitutes. There is another interesting direction

[6] Harsanyi and Selten (1988) considered discretized versions of some games with infinite strategy spaces.

for further research: In the introduction to this chapter we reported about a number of articles in which conditions are derived under which one particular ordering of moves is preferred by both players. One could try to check whether this preferred equilibrium is then in fact also the equilibrium selected by the Harsanyi and Selten equilibrium selection theory. We conjecture that this is not always the case. However, we conjecture that the preferred equilibrium is selected if there are at least two periods at which players can commit. However, in this case the result is driven by the substitution of solutions of the subgames, and not so much by risk dominance.

Appendix

This Appendix provides the details of the proofs that were left open.

Cournot duopoly

Consider the function $G_j(t, b, q_i; q_j)$ that was defined in (7.3.2). Note that only the second term can be positive. This implies that if $G_j(t, b, q_i; q_j) \geq 0$ for some $t < 1$, then $q_j > a_j^N$. For a given q_i, the optimal commitment strategy for player j is obtained by differentiating $G_j(t, b, q_i; q_j)$ with respect to q_j, and equaling the obtained expression to zero. (Note that the second derivative is strictly negative.) This yields

$$q_j^{opt}(t) = \frac{2\alpha_j - \alpha_i + (\alpha_i - 2q_i)t + (1-t)(\alpha_i b - \alpha_j \ln(1+b))}{2(1+t+(1-t)b)}. \tag{A.1}$$

From (A.1) we obtain the optimal commitment strategies in the two tracing procedures. Namely,

$$\begin{aligned}
\tilde{q}_1(t) &= \frac{d_1 v_1 - t v_2}{2(d_1 d_2 - t^2)}, & \hat{q}_1(t) &= \frac{d_2 v_1 - t v_3}{2(d_2^2 - t^2)}, \\
\tilde{q}_2(t) &= \frac{-t v_1 + d_2 v_2}{2(d_1 d_2 - t^2)}, & \hat{q}_2(t) &= \frac{-t v_1 + d_2 v_3}{2(d_2^2 - t^2)},
\end{aligned}$$

where

$$\begin{aligned}
d_i &= 1 + t + (1-t)\bar{z}_i \\
v_1 &= 2\alpha_1 - \alpha_2 + \alpha_2 t + (1-t)(\alpha_1 \bar{z}_2 - \alpha_2 \ln(\bar{z}_2 + 1)) \\
v_2 &= 2\alpha_2 - \alpha_1 + \alpha_1 t + (1-t)(\alpha_2 \bar{z}_1 - \alpha_1 \ln(\bar{z}_1 + 1)) \\
v_3 &= 2\alpha_2 - \alpha_1 + \alpha_1 t + (1-t)(\alpha_2 \bar{z}_2 - \alpha_1 \ln(\bar{z}_2 + 1)).
\end{aligned}$$

In particular, we have that

$$\tilde{q}_i(1) = \hat{q}_i(1) = a_i^N$$

and that

$$\tilde{q}_j(0) = \frac{2\alpha_j - \alpha_i + \alpha_j \bar{z}_i - \alpha_i \ln(\bar{z}_i + 1)}{2(\bar{z}_i + 1)}.$$

Straightforward but very tedious computations yield then

$$\begin{aligned}
G_j(0, \bar{z}_i, \tilde{q}_i(0); \tilde{q}_j(0)) &= \frac{1}{2}(\bar{z}_i + 1)\tilde{q}_j(0)^2 - (1 - \bar{z}_i)(\frac{2\alpha_j - \alpha_i}{3})^2 \\
&\quad - \frac{1}{4}(\bar{z}_i(\frac{3\alpha_j - \alpha_i}{2})^2 - \frac{1}{2}\alpha_i(3\alpha_j - \alpha_i)\ln(\bar{z}_i + 1) + \frac{\alpha_i^2 \bar{z}_i}{4(\bar{z}_i + 1)}) \\
&= \frac{\alpha_j^2}{144(\bar{z}_i + 1)}\left\{18k_j^2(\ln(\bar{z}_i + 1))^2 + 18(\bar{z}_i - 1)(k_j - k_j^2)\ln(\bar{z}_i + 1)\right. \\
&\quad \left. + 2(2 - k_j)^2 + \bar{z}_i(-9 + 18k_j - 18k_j^2) + \bar{z}_i^2(1 - 10k_j + 7k_j^2)\right\}
\end{aligned}$$

where $k_j = \alpha_i/\alpha_j$. Let

$$g(x) = \frac{(4x - 2)^2}{18x^2 - (4x - 2)^2},$$

and

$$\begin{aligned}
f(x) &= 18x^2(\ln(g(x) + 1))^2 + 18(g(x) - 1)(x - x^2)\ln(g(x) + 1) \\
&\quad + 2(2 - x)^2 + g(x)(-9 + 18x - 18x^2) + g(x)^2(1 - 10x + 7x^2)
\end{aligned}$$

Then $g(\alpha_i/\alpha_j) = \bar{z}_i$ and $G_j(0, \bar{z}_i, \tilde{q}_i(0); \tilde{q}_j(0)) > 0$ if and only if $f(\alpha_i/\alpha_j) > 0$. We will show

Lemma 7.5 *$f(x)$ is decreasing on $[2/3, 3/2]$.*

Lemma 7.1 then follows by substitution of $x = 1.0805$ and $x = 1.081$.

Proof. Note that

$$\begin{aligned}
f'(x) &= 36x(\ln(g(x) + 1))^2 \\
&\quad + \left[36x^2 \frac{g'(x)}{g(x) + 1} + 18g'(x)(x - x^2) + 18(g(x) - 1)(1 - 2x)\right]\ln(g(x) + 1) \\
&\quad + 18(g(x) - 1)(x - x^2)\frac{g'(x)}{g(x) + 1} - 4(2 - x) + g'(x)(-9 + 18x - 18x^2) \\
&\quad + g(x)(18 - 36x) + 2g(x)g'(x)(1 - 10x + 7x^2) + g(x)^2(-10 + 14x)
\end{aligned}$$

It is easily verified that the expression between square brackets is positive. Now we can use the fact that

$$\ln(g(x) + 1) \leq g(x) - g(x)^2/2 + g(x)^3/6.$$

This yields

$$\begin{aligned} f'(x) \;\leq\; & \frac{4}{(x^2+8x-2)^6}\Big[-768+17488x-159024x^2+711888x^3-1435844x^4 \\ & +15300x^5+4623516x^6-4540008x^7-5577291x^8+12076593x^9 \\ & -8026610x^{10}+2498964x^{11}-402690x^{12}+30145x^{13}\,\Big] \end{aligned}$$

Call the expression between the brackets $h(x)$. Substitution of $x = y+2/3$ in $h(x)$ yields a polynomial in y of degree 13. We need to show that this polynomial is negative for all $y \in [0, 5/6]$. When we multiply this polynomial with a suitable number (in order to avoid non-integer coefficients), we obtain

$$\begin{aligned} P(y) := 1594323\, h(y+2/3) \;=\; & -11,168,983,684 - 164,908,289,616y \\ & -942,287,374,800y^2 - 2,469,754,612,944y^3 \\ & -2,090,175,577,236y^4 + 3,200,389,079,004y^5 \\ & +6,449,333,520,852y^6 - 1,614,959,549,496y^7 \\ & -9,156,952,284,759y^8 - 3,731,130,088,101y^9 \\ & +1,660,321,004,418y^{10} + 514,122,400,692y^{11} \\ & -225,490,416,300y^{12} + 48,060,866,835y^{13} \end{aligned}$$

To save space we let d_k denote the coefficient of y^k in $P(y)$. Let

$$d = \sum_{k=0}^{5} d_k \left(\frac{6}{5}\right)^{5-k}.$$

By using that $y \in [0, 5/6]$ one verifies that

$$\begin{aligned} P(y) \;&\leq\; dy^5 + d_6 y^6 + d_7 y^7 + d_8 y^8 \\ &\leq\; 10^{12} y^5(-4+7y-y^2-9y^3) < 0 \end{aligned}$$

□

In the remainder of this Appendix we will make use of the fact that we only need to consider the case where $\tilde{G}_1(0) \geq 0$. It follows that we only need to consider cases where $\bar{z}_2 \in [2/7, 7/20]$. In order to prove Lemma 7.2, we need three more lemma's.

Lemma 7.6 *(i)* $\tilde{q}_1(0) = \hat{q}_1(0) > a_1^N$ *and* $\tilde{q}_2(0) > \hat{q}_2(0)$.

(ii) For all $t \in (0,1)$

$$\tilde{q}_1(t) < \hat{q}_1(t) \text{ and } \tilde{q}_2(t) > \hat{q}_2(t).$$

Proof. From inspection of (A.1) it follows immediately that $\tilde{q}_1(0) = \hat{q}_1(0)$. Since $\tilde{G}_1(0) \geq 0$, it must hold that $\tilde{q}_1(0) > a_1^N$.

Fix $t \in [0,1)$ and $q_1 \leq a_1^L$ and consider $f(t, q_1, b) = q_2^{opt}$ as in (A.1) as a function of b. We have that $\partial f / \partial b < 0$ if and only if

$$(\alpha_1+\alpha_1 b-\alpha_2)(1+t+(1-t)b)-(1+b)(2\alpha_2-\alpha_1+(\alpha_1-2q_1)t+(1-t)(\alpha_1 b-\alpha_2\ln(b+1)) < 0.$$

Since the left-hand side of this expression is linear in t, it is enough to establish the inequality for $t = 0$ and $t = 1$. For $t = 0$, the left-hand side is

$$(b+1)(\alpha_1(b+1)-\alpha_2)-2\alpha_2+\alpha_1-\alpha_1 b+\alpha_2\ln(b+1)) = (b+1)(2\alpha_1-3\alpha_2+\alpha_2\ln(b+1)) < 0.$$

For $t = 1$, the left-hand side is

$$\begin{aligned} 2(\alpha_1(b+1)-\alpha_2)-(b+1)(2\alpha_2-2q_1) &\leq 2(\alpha_1(b+1)-\alpha_2)-(b+1)(2\alpha_2-2\alpha_1+\alpha_2) \\ &= 4(\alpha_1-\alpha_2)+4\alpha_1 b-3\alpha_2 b-\alpha_2 < 0 \end{aligned}$$

Hence, $\partial f / \partial b < 0$. Now we take $t = 0$ and $q_1 = \tilde{q}_1(0)$. We have

$$\tilde{q}_2(0) = f(0, q_1, \bar{z}_1) > f(0, q_1, \bar{z}_2) = \hat{q}_2(0).$$

This establishes the proof of (i).

Now suppose that it does not hold for all $t \in (0,1)$ that $\tilde{q}_2(t) > \hat{q}_2(t)$. By continuity of $\tilde{q}_2(t)$, and by (i), there must exist some $t' \in (0,1)$ such that $\tilde{q}_2(t') = \hat{q}_2(t')$. Inspection of (A.1) implies then that also $\tilde{q}_1(t') = \hat{q}_1(t')$. But this implies, by taking $q_1 = \tilde{q}_1(t')$, that

$$\tilde{q}_2(t') = f(t', q_1, \bar{z}_1) > f(t', q_1, \bar{z}_2) = \hat{q}_2(t').$$

This contradicts the definition of t', and hence, establishes that $\tilde{q}_2(t) > \hat{q}_2(t)$ for all $t \in (0,1)$. A final inspection of (A.1) yields then $\tilde{q}_1(t) < \hat{q}_1(t)$ for all $t \in (0,1)$. □

We write $\hat{G}_j(t) = G_j(t, \bar{z}_2, \hat{q}_i(t); \hat{q}_j(t))$ for the gain of player j from committing optimally in the alternative tracing procedure. We will need

Lemma 7.7 *For all $t \in [0,1)$, $\hat{G}_2(t) > \hat{G}_1(t)$.*

Proof. The proof of this lemma involves straightforward but very tedious computations. These computations yield

$$\begin{aligned} &\frac{48(d_2^2-t^2)(G_2(t,\bar{z}_2,\hat{q}_1(t);\hat{q}_2(t))-G_1(t,\bar{z}_2,\hat{q}_2(t);\hat{q}_1(t)))}{(\alpha_1^2-\alpha_2^2)(t-1)} = \\ &(\bar{z}_2+1)(\bar{z}_2-2)(-2\bar{z}_2-1)+t(\bar{z}_2-1)^2(4\bar{z}_2+1) \\ &+t^2(-3-\bar{z}_2-\bar{z}_2^2+4\bar{z}_2^3-2\bar{z}_2^4)/(\bar{z}_2+1) \\ &+6(\bar{z}_2^2-1+t(-1+2\bar{z}_2-2\bar{z}_2^2)+t^2(\bar{z}_2-1)^2)\ln(\bar{z}_2+1) \\ &+3(t-1)(2+2\bar{z}_2+t(3-2\bar{z}_2))(\ln(\bar{z}_2+1))^2 \end{aligned}$$

We call the right-hand side of this equation $g(t, \bar{z}_2)$. It is not difficult to verify that $g(t, \bar{z}_2)$ is concave in t for all $\bar{z}_2 \in [2/7, 7/20]$. This implies that the minimum of $g(t, \bar{z}_2)$ is attained in $t = 0$ or in $t = 1$. Well, substitution yields

$$\begin{aligned} g(0, \bar{z}_2) &= (\bar{z}_2 + 1)((\bar{z}_2 - 2)(-2\bar{z}_2 - 1) + 6(\bar{z}_2 - 1)\ln(\bar{z}_2 + 1) - 6(\ln(\bar{z}_2 + 1))^2) \\ &\geq (\bar{z}_2 + 1)((\bar{z}_2 - 2)(-2\bar{z}_2 - 1) + 6(\bar{z}_2 - 1)\bar{z}_2 - 6\bar{z}_2^2) \\ &= (\bar{z}_2 + 1)(2 - 3\bar{z}_2 - 2\bar{z}_2^2) > 0 \end{aligned}$$

$$\begin{aligned} g(1, \bar{z}_2) &= \frac{9\bar{z}_2 - 6(\bar{z}_2 + 1)\ln(\bar{z}_2 + 1)}{\bar{z}_2 + 1} \\ &\geq \frac{3\bar{z}_2(1 - 2\bar{z}_2)}{\bar{z}_2 + 1} > 0 \end{aligned}$$

for all $\bar{z}_2 \in [2/7, 7/20]$. Hence, $g(t, \bar{z}_2) > 0$, and the lemma is proved. □

Finally, we can prove Lemma 7.2. In fact, we will show the slightly stronger

Lemma 7.8 *Let* $t^* = \max\{t' \in [0,1] | \tilde{G}_1(t) \geq 0$ *for all* $t \in [0, t']\}$. *Suppose that* $t^* > 0$. *For all* $t \in (0, t^*)$ *it holds that*

$$\tilde{G}_2(t) > \hat{G}_2(t) > \hat{G}_1(t) > \tilde{G}_1(t) \tag{A.2}$$

Proof. Suppose it is not true. Then there is some $t \in (0,1)$ such that (A.2) does not hold. Let $\underline{t}$ be the infimum over all such $t \in (0,1)$. By continuity, and the previous lemma, it holds that

$$\tilde{G}_2(\underline{t}) \geq \hat{G}_2(\underline{t}) > \hat{G}_1(\underline{t}) \geq \tilde{G}_1(\underline{t}) \geq 0,$$

where at least one of the inequalities is in fact an equality. Note that, since all the "gain functions" are positive, it must hold that $\tilde{q}_i(\underline{t}), \hat{q}_i(\underline{t}) \in (a_i^N, a_i^L)$, $i = 1, 2$.

The first inequality cannot be an equality: First, we have $\tilde{G}_2(\underline{t}) > G_2(\underline{t}, \bar{z}_1, \tilde{q}_1(\underline{t}); \hat{q}_2(\underline{t}))$, by definition of $\tilde{q}_2(\underline{t})$ and since $\tilde{q}_2(\underline{t}) \neq \hat{q}_2(\underline{t})$ (by Lemma 7.6). Furthermore,

$$\begin{aligned} &G_2(\underline{t}, \bar{z}_1, \tilde{q}_1(\underline{t}); \hat{q}_2(\underline{t})) - G_2(\underline{t}, \bar{z}_2, \tilde{q}_1(\underline{t}); \hat{q}_2(\underline{t})) = \\ &\quad -(1 - \underline{t}) \int_{\bar{z}_1}^{\bar{z}_2} \{u_2(q_1(z); \hat{q}_2(\underline{t})) - u_2(q_1(z); B_2(q_1(z)))\} dz \\ &\quad +(1 - \underline{t})(\bar{z}_2 - \bar{z}_1)(u_2(B_1(\hat{q}_2(\underline{t})); \hat{q}_2(\underline{t})) - N_2) \geq 0 \end{aligned}$$

Finally, we have

$$\begin{aligned} &G_2(\underline{t}, \bar{z}_2, \tilde{q}_1(\underline{t}); \hat{q}_2(\underline{t})) - G_2(\underline{t}, \bar{z}_2, \hat{q}_1(\underline{t}); \hat{q}_2(\underline{t})) = \\ &\quad \underline{t}(\hat{q}_2(\underline{t})(\hat{q}_1(\underline{t}) - \tilde{q}_1(\underline{t})) - \frac{1}{4}(\alpha_2 - \tilde{q}_1(\underline{t}))^2 + \frac{1}{4}(\alpha_2 - \hat{q}_1(\underline{t}))^2) = \end{aligned}$$

$$\underline{t}(\hat{q}_1(\underline{t}) - \tilde{q}_1(\underline{t}))(\hat{q}_2(\underline{t}) - \frac{1}{4}(2\alpha_1 - \hat{q}_1(\underline{t}) - \tilde{q}_1(\underline{t}))) \geq$$
$$\underline{t}(\hat{q}_1(\underline{t}) - \tilde{q}_1(\underline{t}))(a_2^N - \frac{1}{2}\alpha_2 + \frac{1}{2}a_1^N) = 0$$

The second (weak) inequality cannot be an equality either when $\underline{t} > 0$: If $\underline{t} > 0$ it follows that

$$G_1(\underline{t}, \bar{z}_2, \hat{q}_2(\underline{t}); \hat{q}_1(\underline{t})) > G_1(\underline{t}, \bar{z}_2, \hat{q}_2(\underline{t}); \tilde{q}_1(\underline{t}))$$

by definition of $\hat{q}_1(\underline{t})$, and since $\hat{q}_1(\underline{t}) \neq \tilde{q}_1(\underline{t})$ when $\underline{t} > 0$. Furthermore,

$$\begin{aligned}
&G_1(\underline{t}, \bar{z}_2, \hat{q}_2(\underline{t}); \tilde{q}_1(\underline{t})) - G_1(\underline{t}, \bar{z}_2, \tilde{q}_2(\underline{t}); \tilde{q}_1(\underline{t})) = \\
&\quad \underline{t}(\tilde{q}_1(\underline{t})(\tilde{q}_2(\underline{t}) - \hat{q}_2(\underline{t})) - \frac{1}{4}(\alpha_1 - \hat{q}_2(\underline{t}))^2 + \frac{1}{4}(\alpha_1 - \tilde{q}_2(\underline{t}))^2) = \\
&\quad \underline{t}(\tilde{q}_2(\underline{t}) - \hat{q}_2(\underline{t}))(\tilde{q}_1(\underline{t}) - \frac{1}{4}(2\alpha_1 - \tilde{q}_2(\underline{t}) - \hat{q}_2(\underline{t}))) > \\
&\quad \underline{t}(\tilde{q}_2(\underline{t}) - \hat{q}_2(\underline{t}))(a_1^N - \frac{1}{2}\alpha_1 + \frac{1}{2}a_2^N) = 0
\end{aligned}$$

Now consider the case that $\underline{t} = 0$. Note that, by continuity, for small $t > 0$, we still have $\hat{q}_i(t), \tilde{q}_i(t) \in (a_i^N, a_1^L)$. Therefore, we can repeat the whole story with small $t > 0$. This shows that $\underline{t} = 0$ if and only if $\tilde{G}_1(t) < 0$ for small values of $t > 0$. This contradicts the supposition that $t^* > 0$. □

Price setting duopoly with differentiated products

In the remainder of this Appendix we prove Lemma 7.4. Straightforward but very tedious computations show that $D(t) = d_2 t^2 + d_1 t + d_0$, where

$$\begin{aligned}
d_2 &= d(a^4(-7a^4 + 14a^2 - 8) - 4(a^2 - 2)(a^2 - 4)\ln(\frac{2}{2 - a^2})(\ln(\frac{2}{2 - a^2}) - a^2)) \\
d_1 &= d(-4a^6 + 3a^8 + 4(a^2 - 2)(a^2 - 4)\ln(\frac{2}{2 - a^2})(2\ln(\frac{2}{2 - a^2}) - a^2)) \\
d_0 &= d(2(4 - a^2)(a^4 - 2(2 - a^2)(\ln(\frac{2}{2 - a^2}))^2)
\end{aligned}$$

and where

$$d = \frac{(c_2 - c_1)(2 + (a - 1)(c_1 + c_2))(a + 1)}{32a^2(a^2 - 2)(a^2 - 4)} < 0$$

for all $c_1 > c_2$ and all $a \in (0, 1]$. We will show that each of the coefficients is negative, implying that the difference polynomial $D(t)$ only takes negative values for nonnegative values of t.

For $k = 0, 1, 2$ let $f_k(x)$ denote the function such that $f_k(a^2) = d_k/d$. Hence, $f_0(x) = 2(4-x)(x^2 - 2(2-x)(\ln(\frac{2}{2-x}))^2)$. Let $h_0(x) = f_0(x)/(8-2x)$. Now $h_0(0) = 0$ while

$$h_0'(x) = 2x + 2(\ln(\frac{2}{2-x}))^2 - 4\ln(\frac{2}{2-x}).$$

Note that

$$\ln(\frac{2}{2-x}) - 1 = \ln(1 + \frac{x}{2-x}) - 1 \leq \frac{x}{2-x} - \frac{1}{2}(\frac{x}{2-x})^2 - 1 < 0.$$

Hence,

$$\begin{aligned} \frac{1}{2}h_0'(x) &= (\ln(\frac{2}{2-x}) - 1)^2 + x - 1 \\ &\geq \frac{x^3}{9(2-x)^6}(48 - 123x + 120x^2 - 53x^3 + 9x^4) > 0 \end{aligned}$$

for all $x \in (0,1)$. Consequently, $f_0(a^2) > 0$ for all $a \in (0,1)$, and $d_0 < 0$.

We have

$$f_1(x) = -4x^3 + 3x^4 + 4(x-2)(x-4)\ln(\frac{2}{2-x})(2\ln(\frac{2}{2-x}) - x).$$

Note that

$$\ln(\frac{2}{2-x}) = \ln(1 + \frac{x}{2-x}) \geq \frac{x}{2-x} - \frac{1}{2}(\frac{x}{2-x})^2 = \frac{4x - 3x^2}{2(2-x)^2} > 0,$$

and that, consequently,

$$2\ln(\frac{2}{2-x}) - x \geq \frac{x^2 - x^3}{(2-x)^2} > 0.$$

Hence, for all $x \in (0,1)$,

$$\begin{aligned} f_1(x) &\geq -4x^3 + 3x^4 + 4(x-2)(x-4)(\frac{4x-3x^2}{2(2-x)^2})(\frac{x^2-x^3}{(2-x)^2}) \\ &= \frac{x^3}{(2-x)^3}(64 - 8x - 38x^2 + 20x^3 - 3x^4) > 0. \end{aligned}$$

We have

$$f_2(x) = x^2(-7x^2 + 14x - 8) - 4(x-2)(x-4)\ln(\frac{2}{2-x})(\ln(\frac{2}{2-x}) - x).$$

In order to prove that $f_2(x) > 0$ for all $x \in (0,1)$ we will make use of

$$x - \ln(\frac{2}{2-x}) \geq x - (\frac{x}{2-x} - \frac{1}{2}(\frac{x}{2-x})^2 + \frac{1}{6}(\frac{x}{2-x})^3).$$

It follows that

$$f_2(x) \geq \frac{x^3}{3(x-2)^4}(384 - 1096x + 1234x^2 - 690x^3 + 192x^4 - 21x^5) > 0$$

for all $x \in (0,1)$. This completes the proof of Lemma 7.4. □

Chapter 8

Games with Imperfectly Observable Commitment

8.1 Introduction

One of the most important insights in game theory is that the power to commit oneself may confer a strategic advantage: it may be beneficial to constrain one's own behavior in order to induce others to behave in a way that is favorable to oneself. One possibility to commit oneself is to move early: to preempt the others by choosing and communicating the (irreversible) action that one takes before the rivals take their actions. This idea dates back at least to Von Stackelberg (1934) who demonstrated the existence of a "first-mover advantage" in a quantity-setting duopoly. Schelling's (1960) classic *The Strategy of Conflict* generalized Von Stackelberg's initial insight in several dimensions by describing richer commitment tactics as well as illustrating the ubiquity of the phenomenon that in independent decision situations weakness confers strength, that power may result from the power to bind oneself.

Schelling already pointed out that for a commitment to an action to be credible, the commitment must be irreversible, at least reneging should be sufficiently costly. Schelling also stressed that the efficacy of commitment depends on the communication structure of the game. If the opponent is unavailable for messages, or can destroy all communication channels before any communication takes place, being able to commit oneself is of no value. Hence, commitment can be beneficial only if the communication channel is sufficiently reliable. Just how important this latter requirement is has been shown in a recent paper by Kyle Bagwell (1992). Bagwell shows that a precise communication of the commitment is important, that it is vital that there are no ambiguities, that there are no misunderstandings about the action to which the player committed himself. In fact, Bagwell claims that the first-mover advantage is completely eliminated when there is even a *slight* amount of noise associated with the observation of the first-mover's action. Specifically, he shows that, if there is some noise, a pure strategy Nash equilibrium outcome of the game in which one of the players can commit must be a Nash equilibrium outcome of the game in which this commitment possibility is absent. This is a counter-intuitive and striking result and it suggests that a reconsideration of the literature that applies the idea of a "first-mover advantage" might perhaps be required.

The intuition for Bagwell's result can be easily conveyed. Let $g = (A_1, A_2, u_1, u_2)$ be a 2-person normal form game and consider the sequential move game with player 1 moving first. However, assume that player 2 is only imperfectly informed about this commitment. Specifically, if player 1 commits to $a_1 \in A_1$, player 2 receives the signal $a_1' \in A_1$ with probability $p(a_1' \mid a_1) > 0$ where $p(a_1 \mid a_1) \approx 1$. Hence, player 2 is almost perfectly informed about the commitment. The crucial observation, however, is that if player 1 commits to the pure action a_1^*, the signal that player 2 receives is uninformative. Since all information sets of player 2 are reached with positive probability, Bayes' rule dictates that 2 believes that 1 played a_1^* no matter what signal he receives. In equilibrium, player 2 best responds to a_1^* for all possible messages, hence, if 2's best response to a_1^* in g is unique (say it is a_2^*), then 2 will respond with a_2^* no matter what message he receives. However, then, in order to have an equilibrium in the sequential move game, a_1^* should be a best response against a_2^* in g, hence, (a_1^*, a_2^*) must be an equilibrium of g.

As the above paragraph has shown, Bagwell's result is driven by the specific type of imperfection in the communication technology that he assumes. It is not the case that the commitment sometimes is not communicated, it is rather that the opponent with a small probability receives the wrong message. To put it differently, Bagwell's is a model of errors in perception, rather than errors in communication. His result depends on the assumption that if, for example, a seller commits himself to "I do not sell for a price less than \$100", the buyer might interpret this as "I do not sell for less than \$10,000" or as a commitment to "I give the object away for free". We do not want to enter into the debate about whether this is a sensible assumption, although we believe that this specific assumption might explain why Bagwell's result appears counterintuitive at first. It is, however, important to note that the assumption is crucial for the result. If communication errors would take the form as suggested by Schelling (i.e. commitments would not necessarily be communicated to the second mover, but if they would be communicated, they would be communicated without error), then there would not be a lack of robustness of the type that Bagwell notes. The reader can easily verify that in the latter case, as long as the probability that the commitment is received is sufficiently high, a player will commit himself to his Stackelberg strategy. (See Chakravorti and Spiegel (1993)).

As we do not wish to claim that Schelling's modeling of the errors is necessarily better than Bagwell's, we take Bagwell's claim seriously. However, does the theorem that Bagwell proves justify the claim that he makes? Does the result that the pure equilibrium outcomes of the noisy sequential move game coincide with the pure equilibrium outcomes of the simultaneous move game really allow us to conclude that "with even the slightest degree of imperfection in the observability of the first mover's selection (...) the strategic benefit of commitment is totally lost" (Bagwell (1992))? In our opinion such a conclusion would be premature as it would be based on the assumption that only pure strategy Nash equilibria of a game should be taken into consideration. The restriction to pure equilibria, however, is not compelling and the game theory literature has offered no justification for this restriction so far. In fact, the concept of pure strategy Nash equilibrium suffers from the important and well-known drawback of failing to generate a solution for some games. (Existence might be considered the most fundamental property that a solution concept should satisfy.)

In this chapter we take the position that there is no a priori reason to discriminate against equilibria that are not in pure strategies. Consequently, we have to take mixed strategy equilibria into account and this raises the question of which outcomes can be obtained by mixed equilibria of the sequential move game with imperfectly observable commitment. We show that Bagwell's noisy game has a "noisy Stackelberg equilibrium", i.e. a mixed equilibrium that generates an outcome that is close to the Stackelberg outcome and that converges to it as the noise vanishes. Furthermore, we show that there may be other equilibria as well. Hence, Bagwell's game raises the issue of equilibrium selection: If the leader's commitment can only be imperfectly observed, will players coordinate on a pure equilibrium of the simultaneous move game (and, if they do, on which one?) or will they coordinate on the noisy Stackelberg equilibrium? We address this issue in Section 8.4. We argue that, starting from an original situation in which there is uncertainty about which strategies will be played, players will reason themselves to the noisy Stackelberg equilibrium. The argument in this section is motivated by elements from the equilibrium selection theories of Harsanyi and Selten (1988) and from Harsanyi (1993), but the theory that we develop is different from each of these. As we show in Section 8.5, neither the theory of Harsanyi and Selten (1988), nor the theory from Harsanyi (1993) selects the noisy Stackelberg equilibrium in general. The comparison of these various theories gives interesting insights in each of them. Hence, although the main message of this chapter is that there is no immediate need to reconsider the literature that applies the idea of a "first-mover advantage", the chapter may also be read as an exercise in equilibrium selection.

8.2 The noisy commitment game

Notation in this chapter will sometimes differ from that introduced in Section 1.3. This is done to avoid needlessly complicated formulas. For further convenience, this chapter is self-contained.

Let g be a (finite) 2-person game in strategic form. Since below we will mainly be interested in what happens when the players move sequentially rather than simultaneously, we label the players as L (for leader) and F (for follower). $\mathcal{I}$ (resp. $\mathcal{J}$) denotes the set of pure strategies of player L (resp. player F) in g and u_{ij} (resp. v_{ij}) denotes this player's payoff when the strategy pair (i,j) is played. We write $\mathcal{I} = \{1,\ldots,I\}$ and $\mathcal{J} = \{1,\ldots,J\}$. Throughout this chapter we assume that g satisfies the following regularity condition[1]

$$\text{if } (i,j) \neq (k,l), \text{ then } u_{ij} \neq u_{kl} \text{ and } v_{ij} \neq v_{kl}. \tag{8.2.1}$$

This assumption implies that F has a unique best response against each pure strategy i of L. This best response will be denoted by b_i and we write

$$u_i = u_{ib_i}. \tag{8.2.2}$$

Without further loss of generality we assume that

$$u_1 > \max_{i \neq 1} u_i. \tag{8.2.3}$$

[1]Bagwell (1992) restricts himself to the case where player F has a unique best response to any pure action i of player L. He writes that the basic results are most easily reported in this case, from which the reader might be tempted to conclude that his result (Theorem 8.1 in this chapter) is also valid for games that do not satisfy this condition. That conclusion, however, is unwarranted as was shown in Van Damme and Hurkens (1994) (Endnote 1).

Hence, in the sequential move game in which L moves before F and in which F is perfectly informed about the pure action that L has chosen, the unique subgame perfect equilibrium is $(1, b)$ with outcome $(1, b_1)$. (We use b to denote the strategy of F in this game that responds to i with b_i ($i \in \mathcal{I}$).)

We focus our attention on the noisy version of the sequential move game in which F is only imperfectly informed about which action has been chosen by L. To that end, let P be a stochastic matrix defined on the state space $\mathcal{I}$. Hence, $P = (p_{ik})_{i,k\in\mathcal{I}}$ with $p_{ik} \geq 0$ and $\sum_k p_{ik} = 1$ for all i. The interpretation is that F receives the signal "L played k" with a probability p_{ik} in case L actually plays i. Emphasis will be on the situation where the noise, i.e. the probability of receiving the "wrong" signal is small but positive. Writing P^0 for the identity matrix on $\mathcal{I}$ (i.e. $p^0_{ii} = 1$ for all i) we will measure the absolute level of the noise by the distance between P and P^0 and we will write

$$|P| = \max\{|P_{ik} - P^0_{ik}| : i, k \in \mathcal{I}\}. \tag{8.2.4}$$

We will restrict ourselves to the case where any signal can result from any action, i.e.

$$p_{ik} > 0 \text{ for all } i, k \in \mathcal{I}. \tag{8.2.5}$$

Formally then, we consider the extensive form game g^P given by the following rules:

1. player L chooses an action $i \in \mathcal{I}$,

2. chance chooses $k \in \mathcal{I}$ with probability p_{ik},

3. player F learns k and chooses $j \in \mathcal{J}$,

4. player L receives the payoff u_{ij} and F receives v_{ij}.

This game g^P is referred to as the noisy commitment game. Note that the messages (the signals that F receives) are payoff irrelevant. We will denote a (behavioral) strategy of player L (resp. F) in g^P by s (resp. f) and we write $\sigma = (s, f)$ for a strategy combination. Hence, s is a probability distribution on $\mathcal{I}, s \in \Delta(\mathcal{I})$, and f is a map that assigns a probability distribution on $\mathcal{J}$ to each element of $\mathcal{I}$, i.e. $f \in \Delta(\mathcal{J})^{\mathcal{I}}$. We let s_i denote the probability that L chooses i while f_{kj} is the probability that F chooses j in response to the message k. We write $f_k = j$ if $f_{kj} = 1$ and use similar conventions throughout the text. The outcome of the strategy pair $\sigma = (s, f)$ in g^P is the probability distribution $z^P = z^P(\sigma)$ that σ induces on $I \times J$. Hence, we have that

$$z^P(\sigma)_{ij} = s_i \sum_{k=1}^{I} p_{ik} f_{kj} \tag{8.2.6}$$

Note that z^P may involve nontrivial correlation of the players' actions. Player L's (expected) payoff in g^P is written as $u^P(\sigma)$ and F's payoff is denoted by $v^P(\sigma)$, hence

$$u^P(\sigma) = \mathcal{E}(u \mid z^P(\sigma)), \;\; v^P(\sigma) = \mathcal{E}(v \mid z^P(\sigma)) \tag{8.2.7}$$

A pair $\sigma = (s, f)$ is a Nash equilibrium of g^P if s is a best reply against f and f is a best reply against s. Note that because of (8.2.5) there are no unreached information sets in the (extensive form of the) game g^P, hence, any Nash equilibrium is a sequential equilibrium, and in order for f to be a best response against s, it is necessary that f_k is a best response against the posterior beliefs at k induced by s for every message k. By Bayes' rule, this posterior belief that F associates to $i \in I$ after having received the message k is given by

$$\mu_{ik}^{P,s} = p_{ik} s_i / \sum_{\alpha} p_{\alpha k} s_{\alpha}, \tag{8.2.8}$$

so that, for all s with $s_k > 0$

$$\lim_{|P| \to 0} \mu_{ik}^{P,s} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \tag{8.2.9}$$

Hence, if the noise is small and F expects L to choose k with positive probability, then he will attach high probability to the event that L actually played k when he receives the message "k". Assumption (8.2.1) thus implies that F will respond to k with b_k in this case. Lemma 8.1 proves a slightly stronger statement.

Lemma 8.1 *There exists $\varepsilon^* > 0$ such that for all P with $0 < |P| < \varepsilon^*$, all strategy combinations $\tau = (s, f)$ and all $i \in \mathcal{I}$: If $s_i > \sqrt{|P|}$ and f is a best reply against s in g^P, then $f_i = b_i$.*

Proof. The regularity assumption (8.2.1) implies that there exists $\delta < 1$ such that for all $i \in \mathcal{I}$: If player F assigns at least probability δ to L playing i in g, then b_i is the unique best response of F in g. Let ε^* be such that $(1 + \sqrt{\varepsilon^*})^{-1} \geq \delta$.

Now, let P be such that $0 < |P| = \varepsilon < \varepsilon^*$ and let $s \in \Delta(\mathcal{I})$ and $i \in \mathcal{I}$ be such that $s_i > \sqrt{\varepsilon}$. Then we obtain from (8.2.8)

$$\begin{aligned}\mu_{ii}^{P,s} &\geq \frac{p_{ii}s_i}{\varepsilon(1-s_i)+p_{ii}s_i}\\ &= [1+\varepsilon(1-s_i)/p_{ii}s_i]^{-1}\\ &\geq (1+\sqrt{\varepsilon})^{-1}\\ &\geq (1+\sqrt{\varepsilon^*})^{-1}\end{aligned}$$

If f is a best reply against s, then f is necessarily a best reply against the posterior beliefs $\mu_{\cdot i}$ for all i. It, hence, follows from the above inequalities, and the choice of ε^*, that $f_i = b_i$. □

8.3 Equilibria in the noisy commitment game

For the sake of completeness we start by stating (and proving) Bagwell's main result.

Theorem 8.1 (Bagwell (1992)) *The set of pure strategy equilibrium outcomes of g and g^P coincide.*

Proof. Assume (i,j) is a pure strategy Nash equilibrium in g. Then $j = b_i$ and if f is the strategy of F in g^P defined by $f_k = b_i (k \in \mathcal{I})$, then (i,f) is an equilibrium of g^P. It obviously produces the same outcome as (i,j) does. Assume (i,f) is a pure strategy Nash equilibrium in g^P. Since $\mu_{ik}^{P,i} = 1$ for all k, we must have $f_k = b_i$ for all k. Hence, i is a best reply against b_i in g and (i,b_i) is an equilibrium of g with the same outcome as (i,f). □

Theorem 8.1 gives a sufficient condition for an outcome to be an equilibrium outcome of the game g^P. We now give a necessary condition for the case where the noise is small. Write

$$\mathcal{N} = \{(i,b_i): \quad u_i \geq \max_k \min_j u_{kj}\} \tag{8.3.1}$$

for the set of Nash equilibrium outcomes of the game in which player L's commitment is perfectly observed by F. (Note that because of (8.2.1) any Nash outcome has to be pure.) We have that the Nash equilibrium outcome correspondence of g^P is upper-hemi continuous at $P = p^0$.

Theorem 8.2 *Let z^P be an equilibrium outcome of g^P. If $z = \lim_{|P|\to 0} z^P$ exists, then $z \in \mathcal{N}$.*

Proof. The proof follows from regularity assumption (8.2.1) and Lemma 8.1. Let ε^* be as in Lemma 8.1 and for P with $0 < |P| < \varepsilon^*$, let (s^P, f^P) be an equilibrium of g^P with outcome z^P. Assume the limit outcome z to exist. If $i \neq k$, $s_i^P > \sqrt{|P|}$ and $s_k^P > \sqrt{|P|}$, then $f_i^P = b_i$ and $f_k^P = b_k$, hence

$$\lim_{|P|\to 0} u^P(i, f^P) = u_i, \quad \lim_{|P|\to 0} u^P(k, f^P) = u_k.$$

But (8.2.1) implies that $u_i \neq u_k$, hence that $s_i^P s_k^P = 0$ for $|P|$ sufficiently small. The contradiction shows that, for $|P|$ sufficiently small there is at most one $i \in \mathcal{I}$ with $s_i^P > \sqrt{|P|}$. Consequently, we have that $z = (i, b_i)$ for this particular value of i. It is obvious that the inequalities in (8.3.1) must be satisfied. If there would exist $k \neq i$ with $u_i < \min_j u_{kj}$, then L would strictly prefer choosing k above choosing i in g^P for sufficiently small $|P|$. □

Theorem 8.2 implies that, when the noise is small, any equilibrium outcome of g^P is almost pure. This in turn implies that, if g has only mixed equilibria, the equilibrium outcomes of g are disjoint from the limit equilibrium outcomes of the noisy commitment game when the noise vanishes. This shows that a result similar to Theorem 8.1 cannot be proved for a "satisfactory" solution concept, i.e. there does not exist a refinement of the Nash equilibrium concept that generates a nonempty set of solutions for every game for which the equilibrium outcomes of the simultaneous move game coincide with those of the noisy commitment game when the noise vanishes.

It is not true that any Nash equilibrium outcome of the commitment game with perfect observability can be approximated by Nash equilibrium outcomes of games with slight noise: the Nash equilibrium correspondence is not lower-hemi continuous. In the game of Figure 8.1, (B, W) is a Nash outcome of the non-noisy game: It is optimal for L to commit to B if F responds to T with E. However, noise forces F to choose W in response to any signal since W is a dominant strategy. Consequently, only (T, W) can be approximated by equilibrium outcomes of noisy games. (More generally, it follows that g^P has a unique equilibrium outcome in case F has a dominant strategy in g.)

	W	E
T	3,3	0,0
B	2,2	1,1

Figure 8.1.

In this chapter we take the position that there is no a priori reason to discriminate against equilibria that are not in pure strategies. Consequently, we have to take mixed strategy equilibria into account and Theorem 8.2 raises the question of which outcomes of the base game g can be approximated by equilibrium outcomes of the game g^P when the noise vanishes. Theorem 8.3 gives part of the answer: If the noise P is small, there is always an equilibrium that produces an outcome that is close to the Stackelberg outcome, i.e. that is close to the subgame perfect equilibrium outcome of the sequential move game without noise. We will refer to such an equilibrium as a noisy Stackelberg equilibrium.

Theorem 8.3 *The game g^P has an equilibrium $\sigma^P = (s^P, f^P)$ with an outcome z^P that converges to $(1, b_1)$ as $|P| \to 0$.*

Proof. Consider the reduced strategic form $\bar{g}^P$ that results from the strategic form of g^P by eliminating all pure strategies of F that do not prescribe to play b_1 after the signal "1". In this reduced game, player L's expected payoff resulting from playing "1" is approximately u_1 if the noise is small, no matter what F plays. Let $\sigma^P = (s^P, f^P)$ be an equilibrium of $\bar{g}^P$. If $s_i^P > \sqrt{|P|}$ for some $i \neq 1$, then Lemma 8.1 guarantees that $f_i^P = b_i$ provided that $|P| < \varepsilon^*$. However, in this case L's payoff resulting from "i" is approximately u_i, hence, $u^P(i, f^P) < u^P(1, f^P)$, so that player L wants to choose i with probability zero. The contradiction shows that, if $|P|$ is sufficiently small

$$s_i^P \leq \sqrt{|P|} \text{ for all } i \neq 1. \tag{8.3.2}$$

The inequalities (8.3.2) in turn imply that $s_1^P \to 1$ as $|P| \to 0$, hence, (by Lemma 8.1) that at the signal "1" only b_1 is a best response of player F. This shows that σ^P is an equilibrium of g^P if $|P|$ is small. Obviously, the outcome z^P of σ^P converges to $(1, b_1)$ as $|P| \to 0$. □

We have seen two sufficient conditions for limit equilibrium outcomes (Theorems 8.1 and 8.3) and one necessary condition (Theorem 8.2). The necessary condition is not sufficient (Figure 8.1) and the sufficient conditions are not necessary: Also outcomes that are not pure Nash equilibria, nor Stackelberg equilibria of g may be approximated. Consider the game of Figure 8.2 in which L has M as a dominant strategy, so that (M, C) is the unique Nash equilibrium. The Stackelberg equilibrium is (T, W). Consider the noisy commitment game with uniform noise, i.e. $p_{ij} = \varepsilon$ if $i \neq j$ and $p_{ii} = 1 - 2\varepsilon$. It is easily seen that the following strategy combination is an equilibrium of this game: Player L commits to M with probability $\frac{3\varepsilon}{1+\varepsilon}$ and to B with the remaining probability $\frac{1-2\varepsilon}{1+\varepsilon}$; player F responds to signals T and B with E, after signal M he plays C with probability $\frac{2}{4-11\varepsilon}$ and E with probability $\frac{2-11\varepsilon}{4-11\varepsilon}$. The corresponding limit outcome is (B, E).

	W	C	E
T	4,4	0,0	0,0
M	5,0	1,1	5,0
B	0,0	0,0	3,3

Figure 8.2.

We will not attempt to describe exactly which outcomes can be obtained as limits of equilibrium outcomes of the noisy game as the noise tends to zero. Rather we conclude from the Theorems 8.1 and 8.3 that typically there exist multiple limits and, hence, that there exists an equilibrium selection problem. We will attempt to address this selection problem directly and we will propose an argument (an equilibrium selection theory) that actually selects a noisy Stackelberg equilibrium. Our theory incorporates elements from the theory proposed by Harsanyi and Selten (1988) as well as elements from the theory proposed in Harsanyi (1993), however, it differs from these and it may select different outcomes. In particular, neither the theory of Harsanyi and Selten (1988) nor that of Harsanyi (1993) need to select a noisy Stackelberg equilibrium. The next section describes our theory and proves the main result of this chapter, while Section 8.5 discusses the theories of Harsanyi/Selten and Harsanyi.

8.4 Equilibrium selection

The strategy b of player F that prescribes to play the best response b_k against action $k \in \mathcal{I}$ for any signal k is a (weakly) dominant strategy in the (strategic form of) the game where L's commitment is observed perfectly. If there is a slight amount of noise (i.e. $P \neq P^0$), then b is no longer dominant, however, as long as the noise is small, it is quite likely that b is a best response. Specifically, as Lemma 8.1 has shown, if $|P| < \varepsilon^*$ and $s_k > \sqrt{|P|}$ for all k, then b is the unique best response against s in g^P. To put it differently, b is a best response to a set of mixed strategies of player L in g^P that converges to the set of all strategies as $|P| \to 0$. On the basis of these considerations it would seem that L should assign a large (prior) probability to F playing b and, hence, he will be tempted to play his Stackelberg leader strategy "1". However, if $P \neq P^0$ and b_1 is not a dominant strategy in g, then $(1, b)$ is not an equilibrium of g^P, so that a theory that tells player L to play "1" and that tells F to play b is self-destroying. The simple point we make in this section is that, if the players' reasoning process corresponds to the tracing procedure (Harsanyi (1975), Harsanyi and Selten (1988)), then players will finally coordinate on a noisy Stackelberg equilibrium if they start from a prior that assigns sufficient weight to F playing b.

The tracing procedure is a process that gradually adjusts players' plans and expectations until they are in equilibrium. We only describe the mechanics of this procedure, for the motivation and heuristic description of the process we refer to the original sources. Let $\sigma^0 = (s^0, f^0)$ be a mixed strategy combination[2] in g^P. We interpret σ^0 as the players' prior expectations, hence, a priori player F believes that L will play i with probability s_i^0 while L believes that F will play the pure strategy f with probability $f^0(f)$. For $t \in [0,1]$ consider the strategic form g^{P,t,σ^0} defined by

$$u^{P,t,\sigma^0}(i,f) = tu^P(i,f) + (1-t)u^P(i,f^0) \tag{8.4.1}$$

$$v^{P,t,\sigma^0}(i,f) = tv^P(i,f) + (1-t)v^P(s^0,f) \tag{8.4.2}$$

Hence, for $t = 1$ this game coincides with g^P, while for $t = 0$ we have a trivial game in which each player's payoff depends only on his prior expectations. Write $\Gamma^P(\sigma^0)$ for the graph of the equilibrium correspondence, i.e.

$$\Gamma^P(\sigma^0) = \{(t; s, f) | t \in [0,1], (s,f) \text{ is equilibrium of } g^{P,t,\sigma^0}\} \tag{8.4.3}$$

It can be shown that in nondegenerate cases this graph $\Gamma^P(\sigma^0)$ contains a unique distinguished curve that connects the unique equilibrium of $g^{P,0,\sigma^0}$ to an equilibrium (s^1, f^1) of g^P. (See Schanuel et al. (1991) for details.) The (linear) tracing procedure consists of following this curve until its endpoint, and the endpoint $T^P(\sigma^0) = (s^1, f^1)$ is called the linear trace of σ^0 in g^P. The interpretation is that players eventually reason themselves to the equilibrium $T^P(\sigma^0)$ if they start from the prior σ^0. Write $z^P(\sigma^0)$ for the outcome of this linear trace $T^P(\sigma^0)$ in g^P. We have that this outcome is close to the Stackelberg outcome $(1, b_1)$ of g if $|P|$ is small. Formally

Lemma 8.2 *If the prior $\sigma^0 = (s^0, f^0)$ is such that $f^0(b)$ is sufficiently close to one, then* $\lim_{|P| \to 0} z^P(\sigma^0) = (1, b_1)$.

Proof. Let $f^0(b)$ be large enough such that

$$u^{P^0}(i, f^0) < u^{P^0}(1, f^0), \tag{8.4.4}$$

i.e. if player L's commitment is perfectly observed by F, then L strictly prefers to play "1" when F responds with f^0. Note that the regularity condition (8.2.1) implies that

[2]We could equivalently work with behavioral strategies, cf. also (8.5.9).

(8.4.4) holds whenever $f^0(b)$ is sufficiently close to 1. Condition (8.4.4) in turn implies that there exist $\varepsilon > 0$ and $t^* > 0$ such that

$$\text{"1" is strictly dominant for } L \text{ in } g^{P,t,\sigma^0} \text{ if } t < t^* \text{ and } |P| < \varepsilon \tag{8.4.5}$$

Furthermore, by choosing ε sufficiently small we can guarantee that for all $i \neq 1$:

$$\text{if } |P| < \varepsilon, \text{ then } u^P(i, f) < u^P(1, f) \text{ for all } f \text{ with } f_i = b_i \text{ and } f_1 = b_1 \tag{8.4.6}$$

We will restrict ourselves to stochastic matrices P with

$$\sqrt{|P|} \leq t^*/(2I). \tag{8.4.7}$$

Finally, with ε^* as in Lemma 8.1, we assume that

$$|P| < \varepsilon^*. \tag{8.4.8}$$

Let P be such that (8.4.5) - (8.4.8) hold and denote by $\sigma^{P,t} = (s^{P,t}, f^{P,t})$ an equilibrium on the distinguished curve in $\Gamma^P(\sigma^0)$ that connects the unique equilibrium of $g^{P,0,\sigma^0}$ with $T^P(\sigma^0)$. We claim that

$$s_i^{P,t} < 1/(2I) \text{ for all } i \neq 1 \text{ and all } t. \tag{8.4.9}$$

Assume, to the contrary, that there exist some $i \neq 1$ and t such that $s_i^{P,t} \geq 1/(2I)$ and let τ be the smallest t for which an equilibrium of this type can be found. Then $\tau \geq t^*$ in view of (8.4.5). Hence, at $t = \tau$, the total probability that F assigns to L playing i in g^{P,t,σ^0} is at least $t^*/(2I)$, so that (8.4.7), (8.4.8) and Lemma 8.1 guarantee that $f_i^{P,\tau} = b_i$. At the same time we have that

$$s_1^{P,\tau} = 1 - \sum_{i \neq 1} s_i^{P,\tau} > 1 - I/(2I) = 1/2 \geq \sqrt{|P|}$$

so that $f_1^{P,\tau} = b_1$ by the same argument. But now (8.4.5) and (8.4.6) imply that

$$u^{P,\tau,\sigma^0}(i, f^{P,\tau}) < u^{P,\tau,\sigma^0}(1, f^{P,\tau}),$$

hence, $s_i^{P,\tau} = 0$. The contradiction shows that (8.4.9) holds. In particular, we have that $s_1^{P,1} > 1/2$, hence $f_1^{P,1} = b_1$ in view of Lemma 8.1. Applying Lemma 8.1 and (8.4.6) once more we see that, therefore, $s_i^{P,1} \leq \sqrt{|P|}$ for all $i \neq 1$, hence, that

$$\lim_{|P| \to 0} s_1^{P,1} = 1.$$

This completes the proof. □

To complete our argument that players will (or should) coordinate on a noisy Stackelberg equilibrium, we have to give an argument why player L should attach a high prior probability to F playing b. We will borrow such an idea from Harsanyi (1993). Harsanyi proposes that the prior should be based on (should be proportional to) the structural incentive that a player has to use this strategy and he suggests to measure this structural incentive by the size of the stability set.

Formally, Harsanyi proceeds as follows. Let $g = (A_1, A_2, u_1, u_2)$ be a 2-person game and let $S_i = \Delta(A_i)$ be player i's set of mixed strategies. The stability set of $s_i \in S_i$ is the set $St_j(s_i)$ of all mixed strategies of player j against which s_i is a best response. At first it seems natural to measure the structural incentives of a pure strategy a_i by the Lebesgue measure of $St_j(a_i)$, but Harsanyi (1993) shows that this definition would violate certain desirable properties. To circumvent these, Harsanyi first transforms the strategy simplex S_j by the so-called inversion mapping ω_j and he then takes the Lebesgue measure of the transformed set. Formally, ω_j is the mapping from the interior of S_j to the interior of S_j that maps s_j into $\bar{s}_j$ defined by

$$\bar{s}_j(a_j) = s_j^{-1}(a_j) / \sum_{a \in A_j} s_j^{-1}(a). \tag{8.4.10}$$

Hence, Harsanyi measures the structural incentives of player i to use the pure strategy $a_i \in A_i$ by number $\rho(a_i) = \lambda(\omega_j(St_j(a_i)))$ where λ denotes Lebesgue measure. The prior probability that player j then assigns to i playing a_i is proportional to these incentives, hence,

$$p_j(a_i) = \rho(a_i) / \sum_{a \in A_i} \rho(a). \tag{8.4.11}$$

In the special case of our noisy commitment game, we have that the stability set of the strategy b of player F converges to the entire strategy simplex of player L as $|P| \to 0$ and, hence, that the stability set of any other pure strategy converges to a set of measure zero. It follows that the prior of player L, as constructed by using (8.4.10) and (8.4.11) puts almost all weight on the strategy b of player F when $|P|$ is small. Hence, from Lemma 8.2 we can conclude that players will end up in the Stackelberg equilibrium in the limit. Formally, we have proved

Theorem 8.4 *If players construct their prior beliefs by using Harsanyi's (1993) theory of structural incentives and if they update their priors by using the tracing procedure of Harsanyi (1975) to obtain an equilibrium, then, in the limit when the noise vanishes, they will play the Stackelberg equilibrium.*

8.5 Alternative methods of equilibrium selection

8.5.1 Evolutionary and eductive theories

In this section we show that the theories proposed in Harsanyi and Selten (1988) and in Harsanyi (1993) do not necessarily select a noisy Stackelberg equilibrium as the solution of the game with imperfectly observable commitment. The basic reason is that these theories do not consider all equilibria of a game to be eligible as solution candidates. Both theories start by eliminating certain Nash equilibria as candidates. Specifically, equilibria that are considered to have poor stability properties are eliminated. Harsanyi and Selten (1988) first eliminate all equilibria that do not belong to a primitive formation. A formation is a set of strategy pairs that is closed with respect to taking best responses and a formation is said to be primitive if it does not properly contain any other formation. Harsanyi (1993) only considers equilibria that are both proper (Myerson (1978)) and persistent (Kalai and Samet (1984)) as eligible. For generic 2-person games every Nash equilibrium is proper and an equilibrium is persistent if and only if it belongs to a primitive formation.[3] Hence, for generic 2-person games, both theories start from the same set of initial candidates.

	W	E
T	3,3	0,2
B	4,0	1,1

Figure 8.3.

The game displayed in Figure 8.3 may show that the restriction to primitive (persistent) equilibria may eliminate any noisy Stackelberg equilibrium. The game g^P has three equilibria: one corresponds to Theorem 8.1 (with outcome (1,1)), another corresponds to Theorem 8.3 (with outcome close to (3,3)), and there is a third mixed strategy equilibrium. Action B (i.e. the dominant strategy of L in g) is used with positive probability in all three equilibria and the unique best response of player F against B in g^P is to always respond with E. Consequently, $\{(B, EE)\}$ is the unique primitive formation in g^P, hence (B, EE) is the unique persistent equilibrium of this game. Therefore, the theories of Harsanyi/Selten and Harsanyi select the pure equilibrium of g as the solution of

[3]For a proof of the first statement, see Van Damme (1987, Theorem 2.6.2). The second statement follows from the observation that in generic games, a pure strategy that is a best response is a unique best response against an open set of strategies in the neighborhood. See Chapter 2 for further details about the proof.

g^P. These theories confirm Bagwell's claim that slight noise eliminates the commitment power.

The argument used in the above example can be generalized. If one accepts persistency as a selection criterion, one is led to the conclusion that in any game in which the leader has a dominant strategy, slight noise eliminates the benefits of the leader being able to commit himself:

Theorem 8.5 *If player L has a dominant strategy in g, then g^P has a unique primitive formation (resp. persistent retract), viz. the singleton set in which L plays this dominant strategy i and in which F responds with b_i to any signal k.*

Proof. Let $R = R_L \times R_F$ be a persistent retract (resp. primitive formation) and let j be a pure strategy of player L in R_L. Then F has a unique best response against j in g^P, viz. the strategy f with $f_k = b_j$ for all k. Hence, $f \in R_F$. The unique best response of L against f is to play his dominant strategy i from g, hence $i \in R_L$. Let $\bar{f}_k = b_i$ for all k. Then the strict equilibrium $(i, \bar{f})$ belongs to R. Consequently, if R is primitive (persistent), then $R = \{(i, \bar{f})\}$. □

The basic reason why Harsanyi and Selten eliminate equilibria that are not primitive is that such equilibria may have very poor stability properties (cf. Harsanyi and Selten (1988, p. 201) and Harsanyi (1993, footnote 12)). Requiring persistency favors the selection of equilibria that have similar stability properties as strict equilibria, hence, the solution theories of Harsanyi and Selten are biased in favor of the selection of pure equilibria. However, one may very well wonder whether such a bias is justified: The stability property captured by persistency may be relevant in an evolutionary context — where the game is repeatedly played by a large population of players who receive feedback about evolution of the play during the game (see, for example, Chapter 3) — but it is not clear that it has any relevance in the case where the game is played only once and players rely exclusively on deductive personal reflection in order to figure out what to play. At the same time, the theories of Harsanyi/Selten and Harsanyi rely strongly on arguments (such as the tracing procedure) that seem to be particularly relevant in this latter case and that seem irrelevant in the former. Hence, these theories may be criticized for the fact that they mix arguments that are relevant in an evolutionary context with arguments that are relevant in an eductive context. In the following subsections we return to the purely deductive perspective.

8.5.2 Harsanyi's (1993) theory

In this subsection we show that, even in the case where the Stackelberg equilibrium is a strict equilibrium of g^P and, hence, satisfies all of Harsanyi's (1993) eligibility criteria, Harsanyi's theory need not select this Stackelberg equilibrium. The reason is that Harsanyi's theory does not invoke the tracing procedure. Rather, Harsanyi proposes to select as the solution of the game that equilibrium that has the highest prior probability. With the prior probability of a pure strategy as in (4.11), the prior probability of a pure strategy pair a is simply given by

$$p(a) = p_F(a_L)p_L(a_F) \tag{8.5.1}$$

and in the case where only pure equilibria are eligible, Harsanyi selects that equilibrium a^* for which $p(a^*)$ is largest. (At least this is the solution in case the argmax is unique.) The game from Figure 8.4 (in which K is some real positive number) may show that this procedure need not select the Stackelberg equilibrium.

	W	E
T	2,1	0,0
B	0,0	1,K

	WW	WE	EE
T	2,1	$2(1-\varepsilon)$,$1-\varepsilon$	0,0
B	0,0	$1-\varepsilon$,$K(1-\varepsilon)$	1,K

Figure 8.4.

The game g from the left panel of Figure 8.4 is a unanimity game with Stackelberg outcome (2,1). The panel on the right displays (a reduced form of) the game g^P where P involves uniform noise ($p_{ij} = \varepsilon$ if $i \neq j$). We have eliminated the strategy EW for player F in g^P (i.e. the strategy in which F responds to T by E and to B by W) since this is a dominated strategy. Harsanyi indeed suggests to eliminate all dominated strategies before computing the players' structural incentives. The game g^P has three equilibria (T, WW), (B, EE) and a mixed equilibrium with outcome close to (B, E). Only the former two satisfy Harsanyi's eligibility criteria, hence, to compute the Harsanyi solution of the game, we have to compare the prior probabilities of these equilibria. Note that although player L's prior assigns almost all weight to the strategy WE of player F, this prior probability plays no role in this comparison.

Note that the structural incentives for player L to use any of his pure strategies are independent of K: These incentives only depend on player L's own payoff matrix. Furthermore, note that both the prior of T and the prior of B remain bounded away from zero as ε tends to zero. Turning now to the structural incentives of player F, we note that the calculations are simple since, in the 1-dimensional case, the inversion mapping is measure preserving. Hence, the prior probability of a strategy is just the Lebesgue measure of the stability set of that strategy. Straightforward computations show that

$$p_L^\varepsilon(WW) = \varepsilon/(K - K\varepsilon + \varepsilon) \tag{8.5.2}$$

and

$$p_L^\varepsilon(EE) = K\varepsilon/(K\varepsilon + 1 - \varepsilon), \tag{8.5.3}$$

hence

$$\lim_{\varepsilon \downarrow 0} p_L^\varepsilon(EE)/p_1^\varepsilon(WW) = K^2. \tag{8.5.4}$$

It follows that, if K is sufficiently large

$$\lim_{\varepsilon \downarrow 0} p^\varepsilon(T, WW) < \lim_{\varepsilon \downarrow 0} p^\varepsilon(B, EE) \tag{8.5.5}$$

and, hence, that Harsanyi's theory selects the equilibrium (B, EE) in that case. For large values of K, Harsanyi's theory does not select the Stackelberg equilibrium.

8.5.3 Risk dominance and the Harsanyi/Selten theory

An essential ingredient in the equilibrium selection theory from Harsanyi and Selten (1988) is the notion of risk dominance. An equilibrium s is said to risk dominate an equilibrium s' if the tracing procedure, when started at a certain (bicentric) prior $p(s, s')$ ends up at the equilibrium s. (Below we describe how this bicentric prior has to be computed.) Starting from an initial candidate set, Harsanyi and Selten repeatedly eliminate equilibria that are either payoff dominated or risk dominated until finally only one candidate — the solution — remains. We have already seen that the Stackelberg equilibrium need not belong to the initial candidate set, hence, the Harsanyi/Selten theory need not select it. However, in Section 8.5.1 we argued that this elimination step is not convincing. Hence, the question remains whether the Stackelberg equilibrium can be eliminated by considerations of payoff dominance or risk dominance.

Theorem 8.2 implies that the noisy Stackelberg equilibrium cannot be payoff dominated when the noise is small. Any Nash equilibrium outcome of the noisy game converges to a Nash outcome of the game in which the commitment is observed perfectly and among the latter the Stackelberg equilibrium is most preferred by player L. Consequently, it remains to address the question of whether the Stackelberg equilibrium can be risk dominated. We have not been able to resolve the issue in its complete generality, however, for two important subclasses of games — 2×2 games and unanimity games — we can show that the (noisy) Stackelberg equilibrium risk dominates any other equilibrium of g^P when the noise P is small.

To formally define the risk dominance relation we have to describe how the bicentric prior $p(s, s')$ should be computed at which to start the tracing procedure. Harsanyi and Selten have the situation in mind where it is common knowledge among the players that either s or s' is the solution of the game. Each player i will initially assume that his opponent j already knows which of the two is the solution. Player i will assign a subjective probability z_i to the solution being s (and, hence, to j playing s_j) and he will assign the complementary probability $z_i' = 1 - z_i$ to j playing s_j'. After having constructed these beliefs, i will play a best response $b_i(z_i)$ against $z_i s_j + z_i' s_j'$. Player j does not know i's beliefs z_i and, according to the principle of insufficient reason, j will assume that z_i is uniformly distributed on the interval $[0, 1]$. Hence, j will expect i to play the strategy

$$p_j(s, s') = \int_0^1 b_i(z_i) dz_i. \tag{8.5.6}$$

The mixed strategy of player i defined by (8.5.6) describes player j's a priori beliefs which are used to determine the risk dominance relation between s and s'.

Before being able to state the main result of this section, one more definition is needed. We say that $g = (\mathcal{I}, \mathcal{J}, u, v)$ is a unanimity game if (a) $\mathcal{I} = \mathcal{J}$, (b) $u_{ij} = v_{ij} = 0$ for all $i \neq j$, and (c) $u_{ii} > 0$ and $v_{ii} > 0$ for all i. We simplify notation by writing $u_i = u_{ii}$ and $v_i = v_{ii}$ and recall from (8.2.3) that $u_1 > u_i$ for $i \neq 1$. We also write "$\underline{i}$" for the strategy of player F in g^P that prescribes to respond to any signal $k \in \mathcal{I}$ by playing $i \in \mathcal{I}$.

Theorem 8.6 *Let g be a unanimity game. Then the Stackelberg equilibrium $(1,\underline{1})$ risk dominates any other equilibrium of g^P when the noise P is small.*

Proof. We first show that $(1,\underline{1})$ risk dominates any other pure Nash equilibrium of g^P when $|P|$ is small. It suffices to show that $(1,\underline{1})$ risk dominates $(2,\underline{2})$. We first compute the bicentric prior that is used in the risk dominance comparison. Let us first compute the prior p_F of player F. If F plays $z.\underline{1} + (1 - z).\underline{2}$ then the best response of L is

$$b_L^P(z) = \begin{cases} 1 & \text{if } z > \ u_2/(u_1 + u_2) \\ 2 & \text{if } z < \ u_2/(u_1 + u_2) \end{cases} \tag{8.5.7}$$

hence, the prior of F is given by

$$p_F^P(i) = \begin{cases} u_1/(u_1 + u_2) & \text{if } \ i = 1 \\ u_2/(u_1 + u_2) & \text{if } \ i = 2 \end{cases} \tag{8.5.8}$$

Next we compute the prior of player L. If L plays $z \cdot 1 + (1-z) \cdot 2$, then the best response of F depends on the message that F receives and on the size of the noise. However, since the posterior of F puts positive weight only on the actions 1 and 2 of player L, F will respond with either 1 or 2 at each possible message. Furthermore, if the noise is small, then F will respond to the message $i = 1$ (resp. $i = 2$) with the action 1 (resp. 2) for most values of z. Hence,without doing any computations, we may state that player L's prior p_L^P corresponds to a behavioral strategy f^0 of player F that is of the following form:

$$f_{ik}^0 = \begin{cases} \approx 1 & \text{if} \quad i = 1 \quad\quad\ \text{and } k = 1 \\ \approx 1 & \text{if} \quad i = 2 \quad\quad\ \text{and } k = 2 \\ 0 & \text{if} \quad i \notin \{1,2\} \quad \text{and } k \notin \{1,2\} \end{cases} \tag{8.5.9}$$

(f_{ik}^0 is the probability that F responds to signal i with action k.) Now, let the prior $\sigma^0 = (p_F^P, p_L^P) = (p_F^P, f^0)$ be as in (8.5.8), (8.5.9) and let the game g^{P,t,σ^0} be as in (8.4.1), (8.4.2). If t is sufficiently small, then the unique equilibrium $(s^{P,t}, f^{P,t})$ of this game is the best reply against the prior, hence

$$f_{ik}^{P,t} = \begin{cases} 1 & \text{if} \quad i = 1 \quad\quad\ \text{and } k = 1 \\ 1 & \text{if} \quad i = 2 \quad\quad\ \text{and } k = 2 \\ 0 & \text{if} \quad i \notin \{1,2\} \quad \text{and } k \notin \{1,2\} \end{cases} \tag{8.5.10}$$

and, provided that $|P|$ is sufficiently small,

$$s_i^{P,t}(i) = 1 \ \text{ if } \ i = 1. \tag{8.5.11}$$

Hence, in particular, player L chooses the Stackelberg strategy with probability 1 for small t. We claim that, if we move along the distinguished curve in $\Gamma^P(\sigma^0)$ by increasing t, then player F has to switch his strategy before player L does. The argument is simply that, if F does not switch from a strategy as in (8.5.10), then L is facing a convex combination of strategies of type (8.5.9) and (8.5.10), hence, this is just a strategy of type (8.5.9), against which the strategy from (8.5.11) is the unique best response. Hence, as t increases, player F's posterior beliefs put more and more weight on L playing "$\underline{1}$" and gradually F switches to respond with "1" at more and more messages. Such changes in behavior of F however, do not necessitate a change in behavior of L: The strategy from (8.5.11) remains a best response. Consequently, if no equilibrium is reached yet, F will have to change again. Eventually (when t gets close to 1), F's posterior after the message "2" will put so much weight on L playing "1" that F will respond to that message by playing "1" as well. At that point in time we have obtained the equilibrium $(1,\underline{1})$ from g^P and no further adjustments are necessary. Hence, starting at the prior (8.5.8) – (8.5.9), the tracing procedure converges to $(1, \underline{1})$, so that $(1,\underline{1})$ risk dominates $(2,\underline{2})$. Hence, the Stackelberg equilibrium risk dominates any pure equilibrium of g^P.

Next, let s' be a mixed strategy equilibrium of g^P. Theorem 8.2 implies that, if the noise is small, there exists an action $i \in \mathcal{I}$ such that player L plays i with a probability very close to one. If $i = 1$, then $(1,\underline{1})$ is the unique equilibrium of g^{P,t,σ^0} for all t. If $i \neq 1$, then the proof follows exactly the same line as above: Player L plays "1" for each value of t and player F switches several times until he finally responds to all messages by playing "1". □

Our final result is

Theorem 8.7 *If g is 2×2 game and $|P|$ is small, then g^P has one equilibrium that risk dominates all other equilibria and the outcome generated by this risk dominant equilibrium converges to the Stackelberg outcome $(1, b_1)$ as $|P| \to 0$.*

Proof. The result follows from Theorem 8.3 in case player F has a dominant strategy in g (g^P has only one equilibrium in this case). Hence, assume that F does not have a dominant strategy. Without loss of generality assume $b_1 = 1$ and $b_2 = 2$. In case g does not have any pure equilibria, the result again follows from Theorem 8.3 since g^P has a unique equilibrium in this case. (The unique best response of F against strategy i of player L is to respond with i to any message, but then L's best response is to play $j \neq i$.) There are three cases left to consider:

(i) (1,1) is the unique pure equilibrium of g.

(ii) (2,2) is the unique pure equilibrium of g.

(iii) both (1,1) and (2,2) are pure equilibria in g.

The first case is easy: It can be resolved by iterative elimination of strictly dominated strategies. (It should be obvious from the description of risk dominance on the preceding pages that strategies that are iteratively strictly dominated cannot influence the risk dominance relationship.) The strategy "21" of player F (play $k \neq i$ in response to i for $i = 1, 2$) is strictly dominated and once this strategy has been eliminated, the strategy 1 becomes strictly dominant for player L. (Note that action 1 is dominant for L in g in case (i).) The third case is very much like the case considered in Theorem 8.6 and the proof proceeds along the same lines. We leave the details to the reader. In case (ii), g^P has three equilibria, viz. a mixed equilibrium with outcome close to (1,1), a mixed equilibrium with outcome close to (2,2), and the pure equilibrium $(2, \underline{2})$. We have to show that the first equilibrium risk dominates the latter two. The proof follows from Lemma 8.2. Namely, consider the bicentric prior p_L^P of player L in game g^P relevant for the comparison between the noisy Stackelberg equilibrium and the pure equilibrium $(2, \underline{2})$. The reader easily verifies that

$$\lim_{|P|\to 0} p_L^P(b) = 1, \tag{8.5.12}$$

since the strategy b of player F (with $b_i = i$ all i) is a best response to the noisy Stackelberg equilibrium and is "almost" a best response to the pure equilibrium. Hence, it follows from Lemma 8.2 that the noisy Stackelberg equilibrium risk dominates the pure Nash equilibrium. To show that this equilibrium also risk dominates the third mixed equilibrium, we note that the strategy b of player F is the unique best response against a strict convex combination of the two mixed equilibrium strategies of player L in g^P. Hence, in this case the prior satisfies $p_L^P(12) = 1$ and the conclusion again follows from Lemma 8.2. □

Although we conjecture that the result from the Theorems 8.6 and 8.7 can be generalized to other classes of games, we have to admit that we have not been able to find a general proof. (We do not have a counterexample either.) However, we note that applying the tracing procedure can be rather complex, so that a multilateral procedure as that in Section 8.4 – in which the tracing procedure is applied only once – might be preferable to a theory in which one is forced to make a rather large number of bilateral comparisons. Furthermore, in order to apply the Harsanyi/Selten theory one has to first compute all (primitive) equilibria of the game. We were able to prove Theorem 8.4 without knowing this set of all equilibria.

8.6 Conclusion

From the fact that any pure Nash equilibrium of a 2-person simultaneous move game is also a pure Nash equilibrium outcome of the sequential move game in which the follower can only observe imperfectly the action to which the leader committed himself (Theorem 8.1 in this chapter), Kyle Bagwell concluded in his 1992 paper that slight noise eliminates any first mover advantages. In the concluding section of his paper, Bagwell writes

> "For applied theorists, the key message of the paper is that the many predictions derived from models with commitment may require reconsideration. Apparently these predictions are valid only for settings in which the committed action is in fact *perfectly* observed by subsequent players. This requirement is quite stringent, and it would seem to be violated in a number of real-world settings to which popular commitment models are thought to apply" (Bagwell (1992, p. 9), emphasis in original).

While we agree with the observation that the assumption of perfect observability is stringent, we disagree with the statement that this assumption is crucial. In fact, we would claim that this chapter shows that the assumption is inessential. Not only have we shown that the noisy game analyzed by Bagwell has always an equilibrium outcome that is close to the subgame perfect equilibrium of the game in which the commitment can be observed perfectly (Theorem 8.3), we have also given several arguments for why players should coordinate on this particular equilibrium (Theorems 8.4, 8.6 and 8.7). In addition, we have remarked that the structure of the noise as assumed by Bagwell is somewhat peculiar and that other specifications, which are, perhaps, more natural and which are closer to Schelling's original ideas (Schelling (1960, p. 149)) also allow the conclusion that the assumption of perfect observability is inessential. Hence, we do not see any need to reconsider the fundamental game theoretic insight that the power to commit oneself may be beneficial.

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Samenvatting (Summary in Dutch)

Games, Rules, and Solutions (Spelen, Regels en Oplossingen) is een proefschrift over niet-coöperatieve speltheorie. Het heeft tot doel de consequenties te onderzoeken van het formeel opnemen van bepaalde strategische zetten (zoals "commitments" en communicatie) in de regels van het spel. Het onderzoekt tevens de werking van enkele oplossingsconcepten die recentelijk geïntroduceerd werden.

Hoofdstuk 1 is inleidend.

Hoofdstuk 2 begint met een korte discussie van oplossingsconcepten die als oplossing van een spel een (of meerdere) verzamelingen van strategieën voorschrijven. Verschillende van dergelijke oplossingsconcepten worden gedefinieerd. De nadruk zal liggen op verzamelingen van strategieën die gesloten zijn onder de toevoeging van (een soort) beste antwoorden. In het bijzonder zullen we curb[1], curb*, robuuste en persistente verzamelingen definiëren, en ook zogenaamde primitieve, primitieve*, robuuste en persistente formaties. We leiden enkele eigenschappen van deze concepten af waarvan we in latere hoofdstukken gebruik zullen maken. We geven verscheidene voorbeelden waaruit het subtiele onderscheid tussen de genoemde concepten zal blijken. Dit hoofdstuk wordt besloten met een discussie van twee andere "verzamelings-oplossingsconcepten".

Het is belangrijk te weten of een oplossingsconcept relevant is in een eductieve of een evolutionaire context. Hoofdstuk 3 toont aan dat curb en persistente verzamelingen relevant zijn in een evolutionaire context. Er wordt een dynamisch leerproces gepresenteerd dat alsvolgt getypeerd kan worden: Spelers hebben een beperkt geheugen en spelen beste antwoorden tegen verwachtingen die gevormd zijn op basis van de strategieën die recentelijk gebruikt werden. We laten zien dat de spelers uiteindelijk strategieën zullen spelen uit een minimale curb verzameling. Dit resultaat blijft, ook als sommige spelers geen beste antwoorden spelen maar andere spelers nabootsen. Wanneer de spelers enigszins onzeker zijn over de strategieën die door de andere spelers gebruikt worden, convergeert het proces niet naar een curb verzameling, maar naar een curb*, robuuste of persistente

[1] De term *curb* is de afkorting van de Engelse vertaling van 'gesloten onder rationeel gedrag'.

verzameling, afhankelijk van hoe de onzekerheid gemodelleerd wordt.

In Hoofdstukken 4 en 5 wordt onderzocht wat de consequenties zijn als we toelaten dat sommige spelers wat geld mogen verbranden voordat een spel gespeeld wordt. Op het eerste gezicht lijkt deze mogelijkheid tot het verbranden van geld irrelevant. Het lijkt onwaarschijnlijk dat iemand gebruik zal maken van de gelegenheid, en, dientengevolge, zal de uitkomst van het spel waarschijnlijk niet beïnvloed worden. Het blijkt echter dat de mogelijkheid van het verbranden van geld wiskundig gezien equivalent is met het toelaten van kostbare communicatie, en dat het wel degelijk de uitkomst van het spel kan beïnvloeden. In Hoofdstuk 4 bekijken we een spel met een willekeurig aantal spelers onder wie sommigen de mogelijkheid hebben om eerst wat geld te verbranden. We laten zien dat de spelers die geld kunnen verbranden hun geprefereerde uitkomst krijgen in alle strategieën in de minimale curb (of curb*) verzameling, zonder dat ze echt geld hoeven te verbranden. Dit resultaat geldt ook voor de persistente verzamelingen, maar alleen in het geval van een spel met twee spelers. In Hoofdstuk 5 bestuderen we de mogelijkheid tot het verbranden van geld in een spel met asymmetrische informatie. Er zijn twee spelers, een Zender en een Ontvanger. De Zender heeft privé informatie over zijn type terwijl de Ontvanger een actie moet kiezen. De opbrengst voor de spelers hangt zowel af van het type van de Zender als van de actie gekozen door de Ontvanger. Voordat de Ontvanger een actie kiest, zendt de Zender een kostbare boodschap naar de Ontvanger. We laten zien dat elk type Zender zijn geprefereerde actie krijgt in elke strategieëncombinatie uit een minimale curb (of curb* of persistente) verzameling.

Het is al langer bekend dat de volgorde waarin spelers hun beslissingen nemen erg belangrijk is. In de meeste modellen is deze volgorde vastgelegd door de regels van het spel. Soms is het voordelig om als eerste een zet te kunnen doen. In dat geval zal iedere speler zich willen verplichten om een bepaalde actie te kiezen zodat de andere spelers zich voor een gedongen feit geplaatst zien en zich wel aan moeten passen.

Hoofdstuk 6 onderzoekt welke evenwichten van een bi-matrix spel overblijven wanneer beide spelers de gelegenheid gegeven wordt om zich te binden aan een bepaalde actie. We bestuderen daarvoor een model met een endogene tijdsbeslissingsstructuur. In dit model moeten de spelers de afweging maken tussen enerzijds, zich binden aan een actie (en daarmee de ander dwingen om een best antwoord te spelen) en anderzijds, afwachten en daardoor in de gelegenheid zijn om optimaal te reageren indien de ander zich gebonden heeft. We laten zien dat een gemengd evenwicht van het oorspronkelijke spel alleen dan overblijft in het spel met een endogene tijdsbeslissingsstructuur als het "commitment robust" is, dat wil zeggen, als niemand een prikkel heeft om de eerste zet te doen. We laten echter ook zien dat elk puur evenwicht van het oorspronkelijke spel een perfecte evenwichtsuitkomst is van het uitgebreide spel. Als we echter alleen curb* of persistente evenwichten accepteren als de oplossing van het uitgebreide spel, dan kunnen we concluderen dat alleen de evenwichten die "commitment robust" zijn overblijven.

Hoofdstuk 7 analyseert enkele economische spelen met een uniek evenwicht dat niet "commitment robust" is. Het spel waarin de tijdsbeslissingsstructuur endogeen bepaald wordt, heeft nu drie pure deelspel-perfecte evenwichten: Er wordt ofwel een Stackelberg evenwicht gespeeld, waarin één speler zich bindt en de ander afwacht (twee mogelijkheden), of er wordt een evenwicht gespeeld waarin beide spelers zich binden zich aan het evenwicht van het oorspronkelijke spel. Om een unieke selectie te maken tussen deze drie evenwichten passen we delen van de evenwichtsselectietheorie van Harsanyi en Selten (1988) toe. We laten zien dat de beide Stackelberg evenwichten het derde evenwicht risico-domineren. Om tot een definitieve selectie uit de twee Stackelberg evenwichten te komen, beperken we ons tot drie economisch interessante spelen. Deze zijn (1) Cournot competitie (in hoeveelheden), (2) Bertrand competitie (in prijzen) en (3) individuele bijdragen aan een publiek goed. We nemen aan dat de spelers verschillende kosten hebben. In elk van de drie spelen onder beschouwing laten we zien dat het Stackelberg evenwicht, waarin de speler met de laagste kosten zich bindt, risico dominant is.

In Hoofdstukken 6 en 7 namen we aan dat een speler zich kan binden tot een actie, en dat de andere speler perfect geïnformeerd wordt over die actie. In Hoofdstuk 8 analyseren we de situatie waarin slechts één speler (de leider) zich kan binden. Maar de actie van de leider wordt niet perfect geobserveerd door de andere speler (de volger). Met een kleine kans observeert de volger een andere actie. Bagwell (1992) claimde namelijk dat het voordeel van het zich kunnen binden, helemaal verdwijnt als er ook maar de geringste imperfectie bestaat in de observatie van de volger. In dit hoofdstuk laten we zien dat deze claim cruciaal afhangt van het feit dat Bagwell zich beperkt tot pure evenwichten. Namelijk, het spel dat geanalyseerd werd door Bagwell heeft altijd een evenwicht in gemengde strategieën waarvan de uitkomst dicht in de buurt ligt van de uitkomst van het Stackelberg evenwicht van het spel waarin de actie van de leider perfect wordt geobserveerd door de volger. We introduceren een nieuwe theorie van evenwichtsselectie. Deze theorie combineert elementen van de theorie van Harsanyi en Selten (1988) met die van Harsanyi (1993). Als de kans dat een verkeerde actie wordt geobserveerd niet te groot is, dan selecteert onze theorie het Stackelberg evenwicht.

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